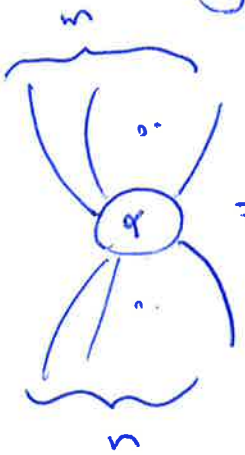
Introduction to ZX calculus

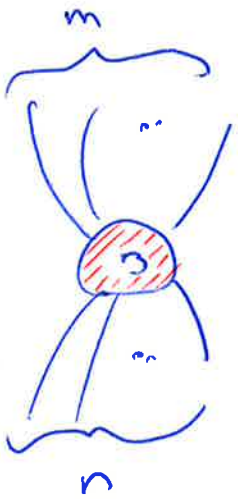
* Building blocks



$= |0\ldots0\rangle\langle 0\ldots 0| + e^{iq} |1\ldots1\rangle\langle 1\ldots 1|$

~ δ-tensor in Z basis

~ Z spider

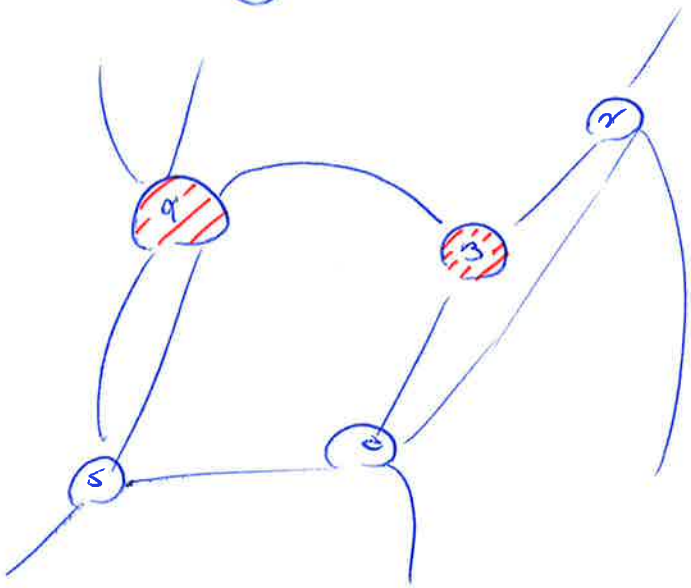


$= |+\ldots+\rangle\langle +\ldots +| + e^{in} |-\ldots-\rangle\langle -\ldots -|$

~ δ-tensor in X basis

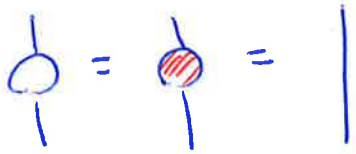
~ X spider

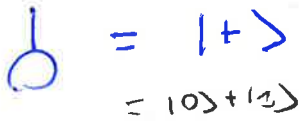
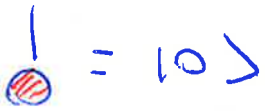
* ZX diagram = 'web of spiders'



only connectivity matters!

Examples

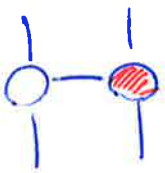
→  (identity) $= |0\rangle\langle 0| + |1\rangle\langle 1|$
$= |+\rangle\langle +| + |-\rangle\langle -|$

→  $= |+\rangle = |0\rangle + |1\rangle$  $= |0\rangle = |+\rangle + |-\rangle$

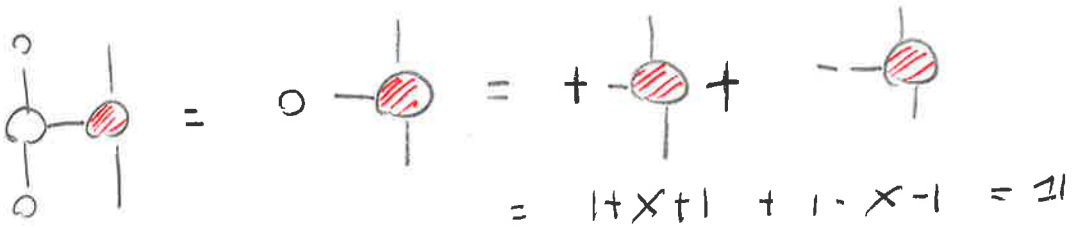
→ Bell state  $= |00\rangle + |11\rangle$
$= |++\rangle + |--\rangle$

→ GHZ state  $= |000\rangle + |111\rangle$

→ CNOT gate

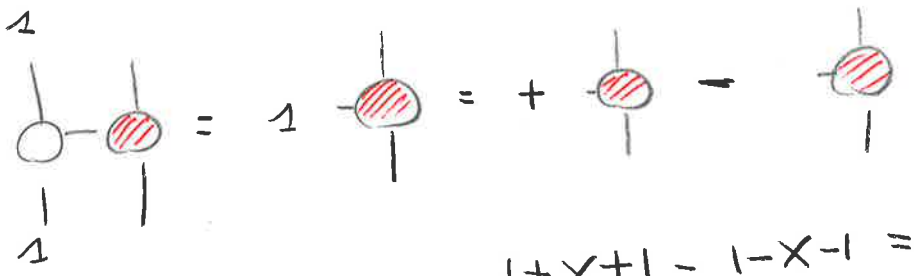


Verify



$= |+\rangle\langle +| + |-\rangle\langle -| = I$

Using $|0\rangle = |+\rangle + |-\rangle$

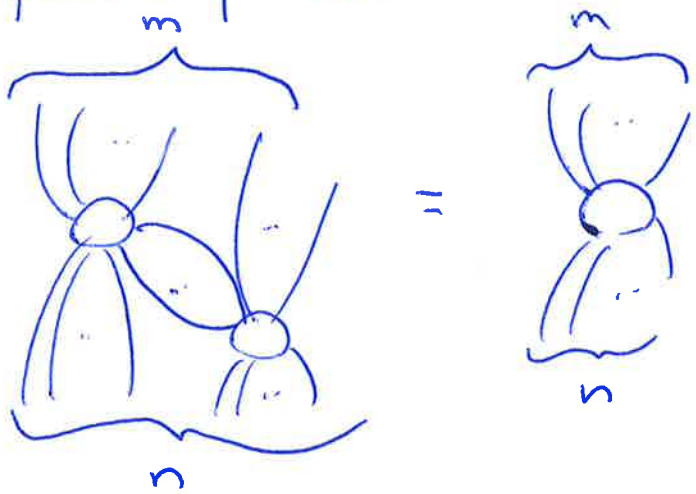


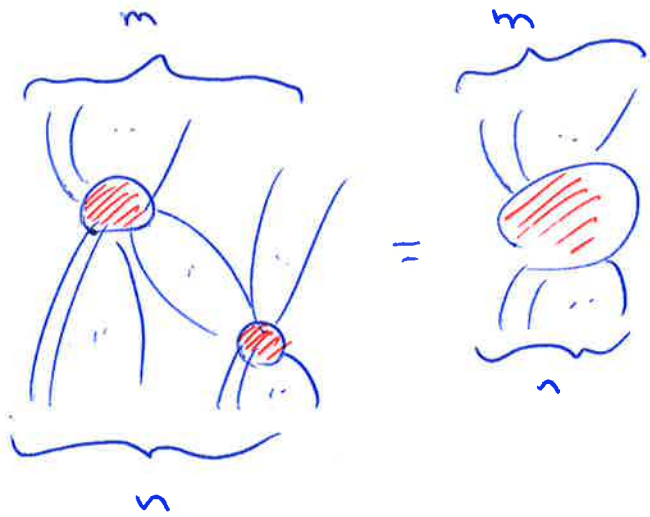
$= |+\rangle\langle +| - |-\rangle\langle -| = X$

Using $|1\rangle = |+\rangle - |-\rangle$

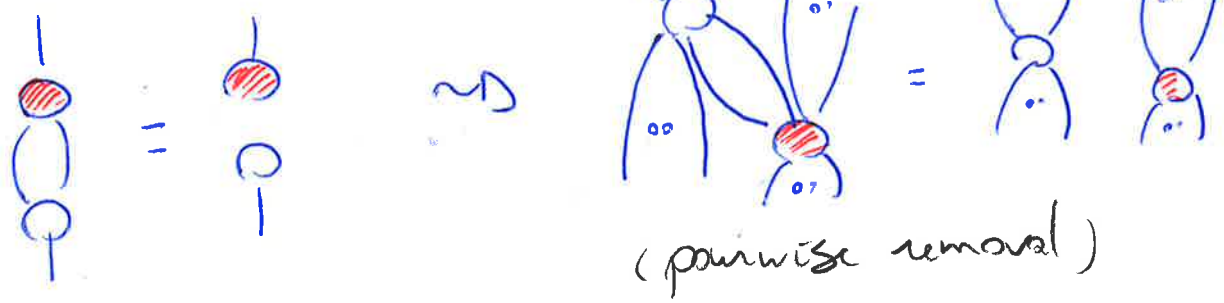
* Graphical rules
(no phases)

• Spider fusion





• Hopf rule



(pairwise removal)


$= |+X+|+|+|-X--|$


$= |0X00| + |1=X121|$

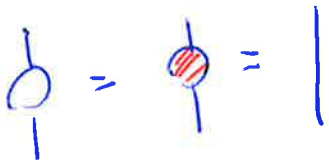
$\rightsquigarrow (|+X+|+|+|-X--|)(|00X01| + |12=X21|)$

$= |+X+|00X01| + |+X+|12=X21|$
$+|-X--|00X01| + |-X--|12=X21|$

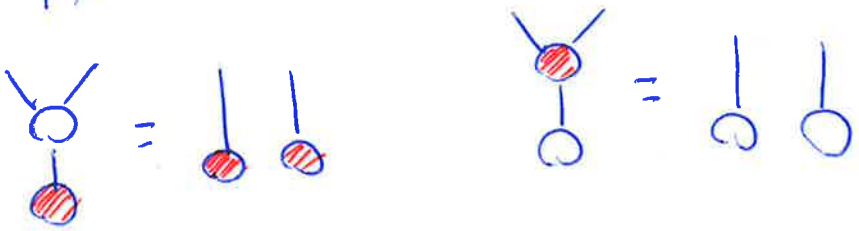
$= |+X01| + |+X21| + |-X01| + |-X21|$

$= (|+>+|->)(<01| + <21|)$

- **Identity rules**

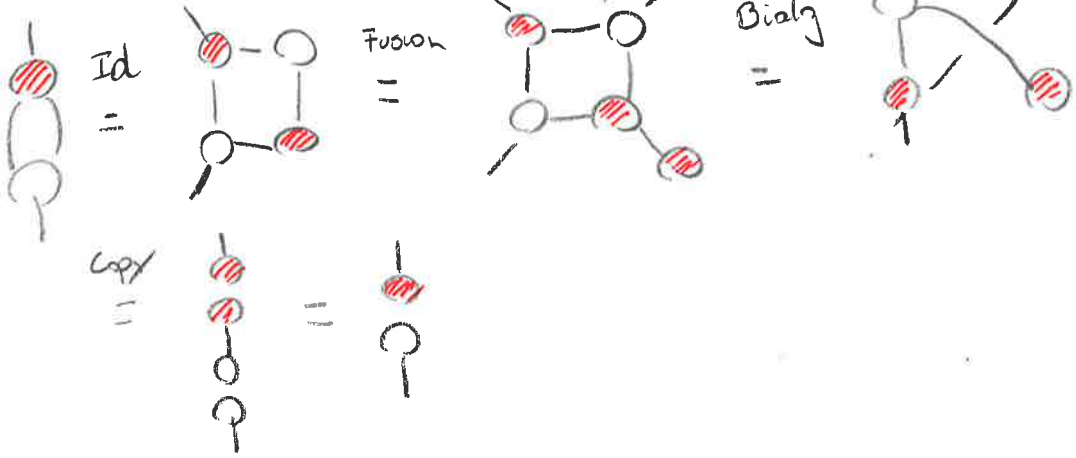


- **Copy rules**

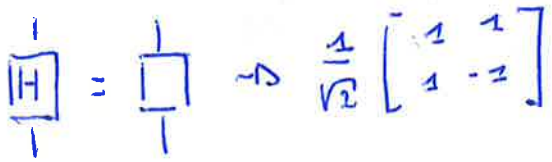


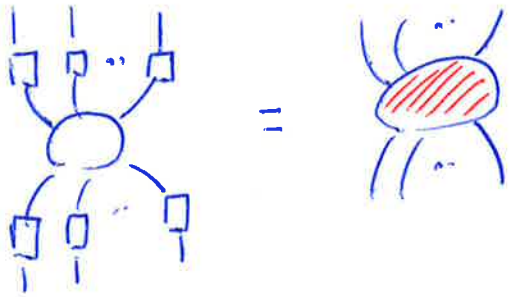
- **Bialgebra rules**

Implies Hopf rule



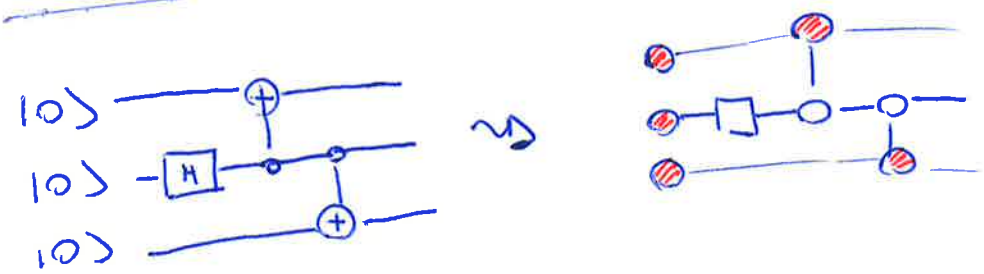
* Colour change
↝ introduce Hadamard matrix



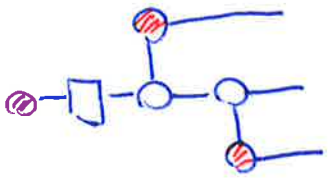


* Examples

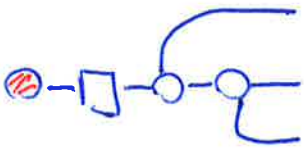
GHZ state preparation



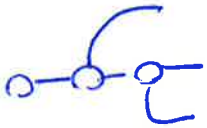
Fusion
↝



Identity
↝

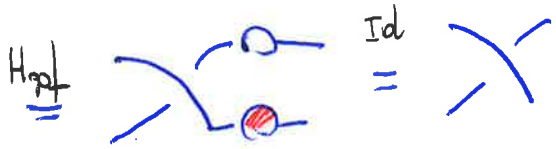


Had
↝

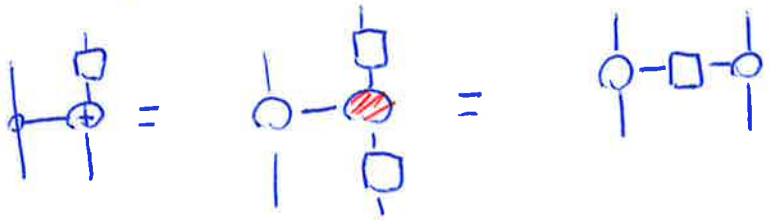


Fusion
↝

3 CNOTs make a SWAP



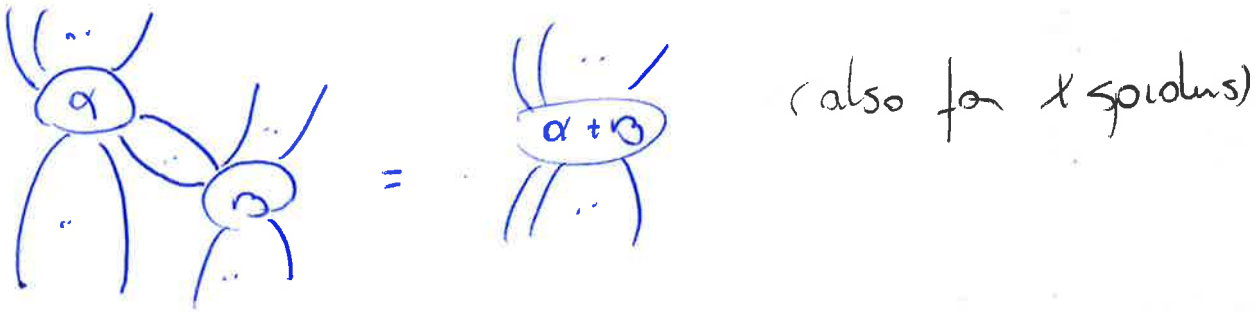
CZ from CNOT



(highlights symmetry + diagonal in Z basis)

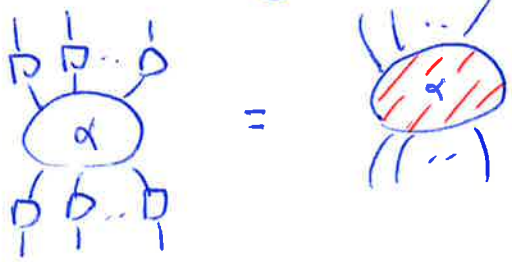
* Graphical rules w/ phases

- Spider fusion

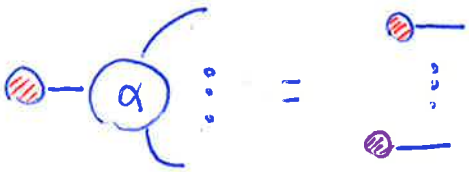


(also for X spiders)

- Color change

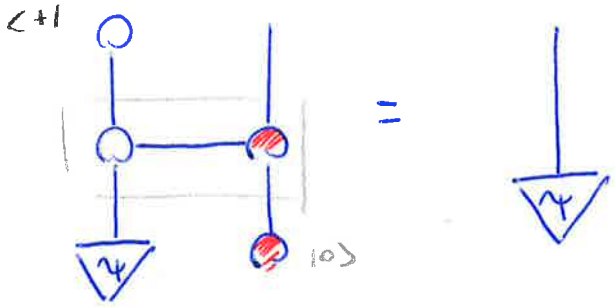


- Copy

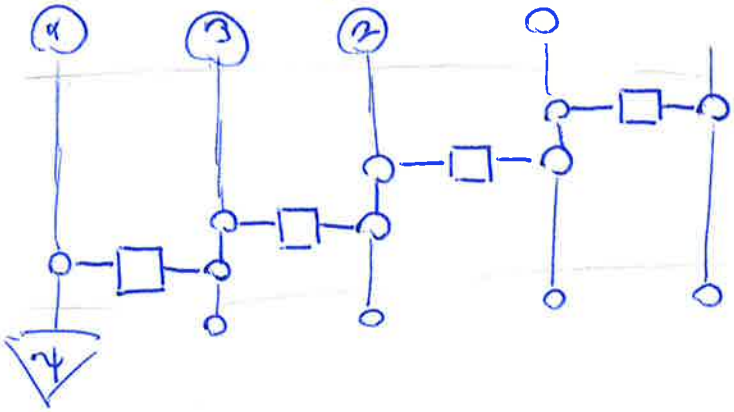


* Application: measurement-based quantum computing

↝ pass state



↝ state preparation

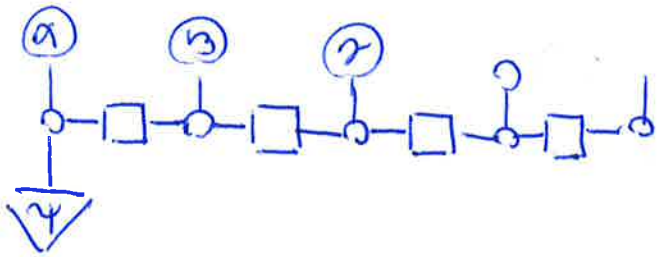


↝ measurements in different bases

↝ CZ's

↝ $|\psi\rangle |++ ++\rangle$

=



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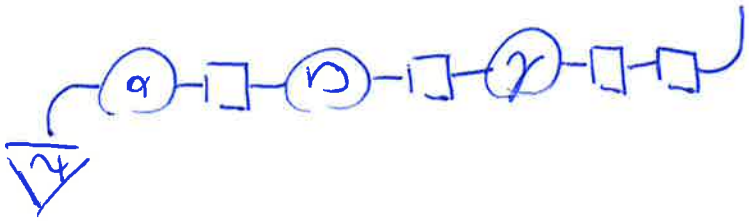

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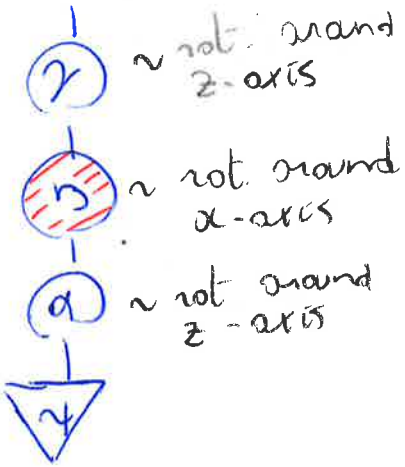


Euler decomposition!

* Application: Clifford gates

→ 'Clifford fragment of ZX': only phases that are an integer multiple of $\pi/2$

→ graphical rules are complete!

[= any calculation that can be done algebraically can be performed using the graphical rules introduced so far] [Does not hold for universal gate sets]

Examples

Z = 

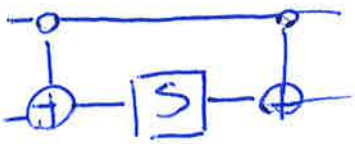
X = 

S = 

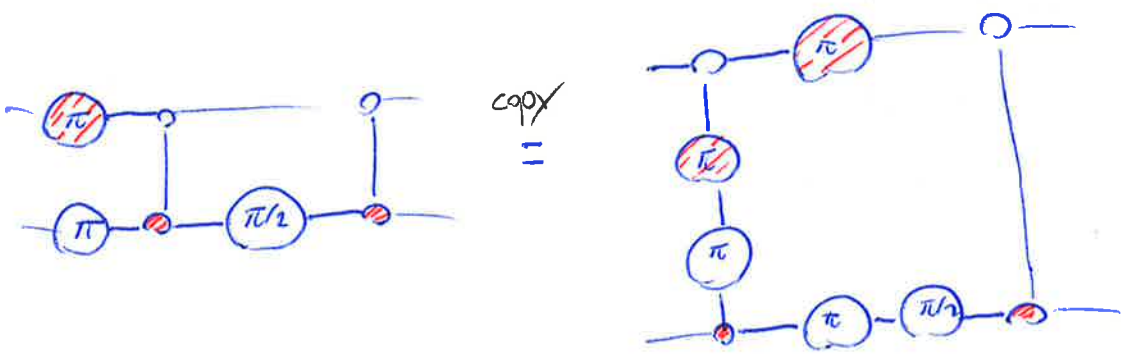
Clifford: maps Paulis to Paulis

⇒ can use ZX rules to 'pull' Paulis through circuit

Example



, action on $X \otimes Z$?


copy
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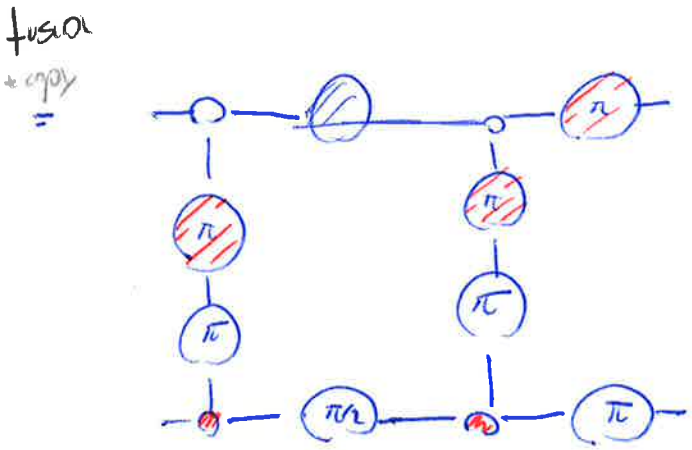
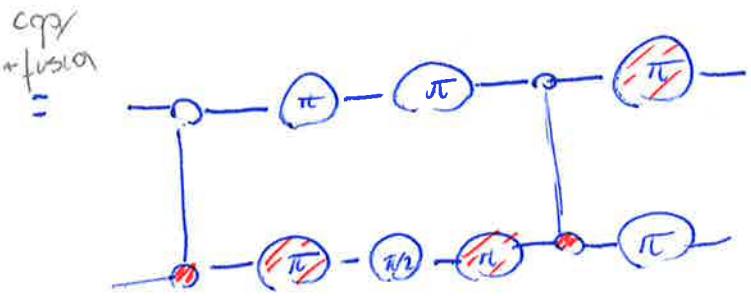

fusion
+ copy
=

[Now use  ]
copy


copy
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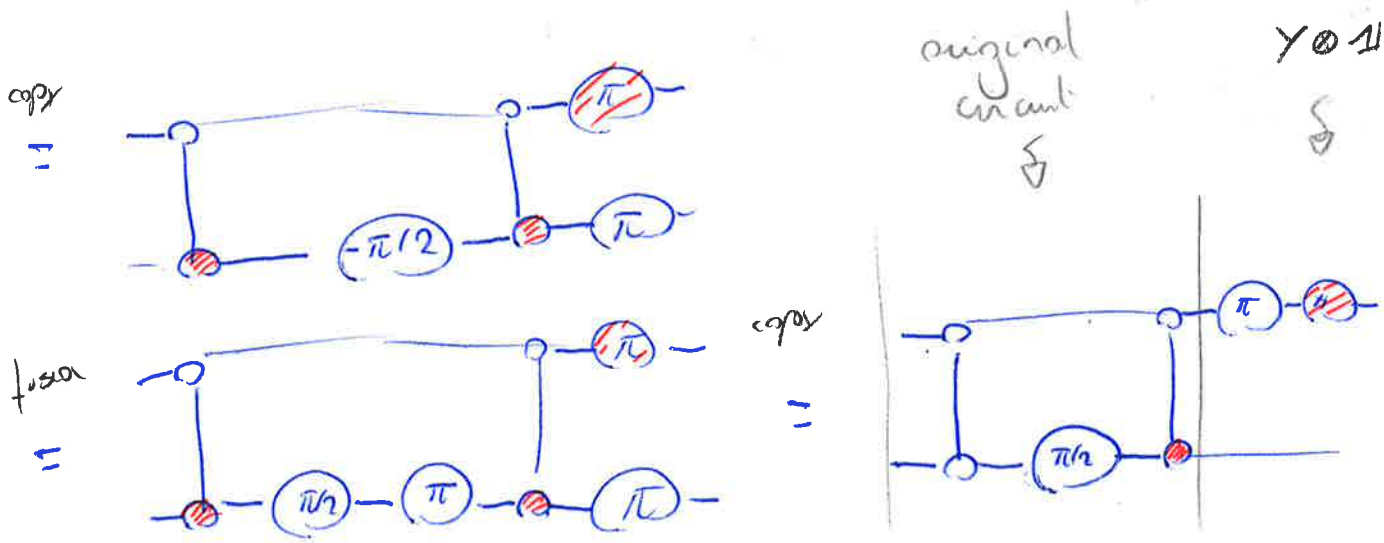
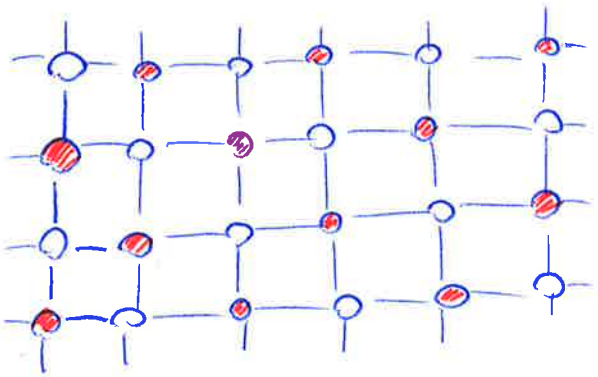
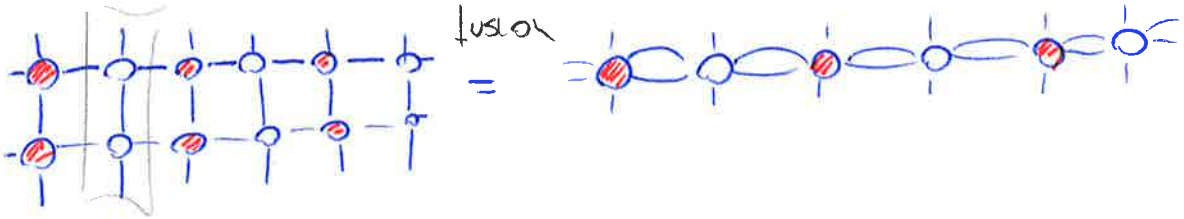

original circuit
Y ⊗ 1
copy
=
-π/2
fusion
=
copy
=
π/2

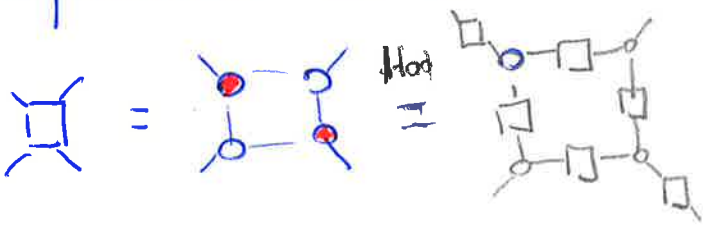
* Application: Dual-unitary circuits



⇝ unitarity follows from



⇝ space-time self-dual by construction
⇝ equivalent to brickwork circuit with



⇝ Kicked Ising Model