Group Meeting

Date :- 22-04-2024

Hierarchial Dual unitary Permutation Gates

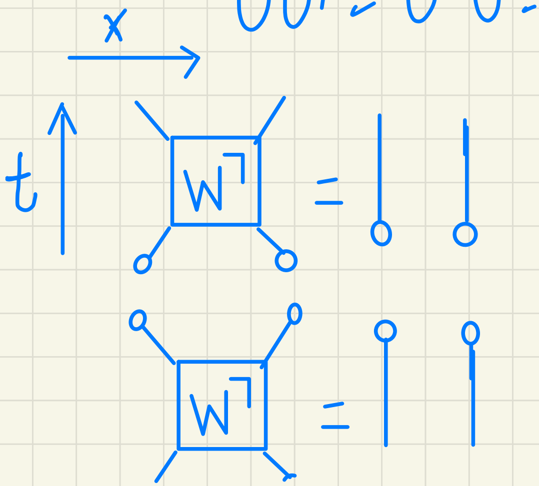
By

Suhail Ahmad Rather

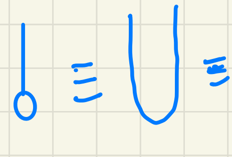
Let us first define hierarchial dual unitarity:

0) <u>Unitarity of U</u>: $U: \mathbb{C}^q \otimes \mathbb{C}^q \mapsto \mathbb{C}^q \otimes \mathbb{C}^q$

$$UU^\dagger = U^\dagger U = \mathbb{1}$$

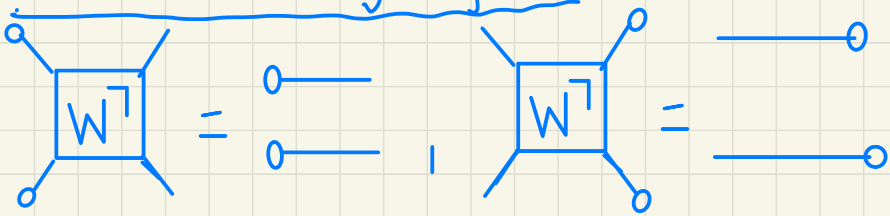


; $W = U \otimes U^*$ complex conjugate of U

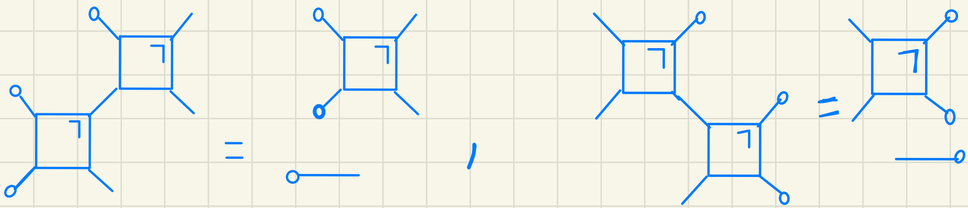


$$= \frac{1}{\sqrt{q}} \sum_{k=0}^{q-1} |k\rangle |k\rangle$$

1) <u>Dual unitarity of U</u>:



2) Level-2 hierarchy; $\mathcal{L}_2$ gates:



Similarly, one can define higher levels of hierarchy by adding more gates. Here we focus mainly on Level-1 (Dual unitaries) and Level-2 gates. From the definitions of hierarchical gates, it is easy to see the following inclusion:

$$\mathcal{L}_{k-1} \subseteq \mathcal{L}_k \quad , \quad k \geq 2$$

we denote by $\bar{\mathcal{L}}_k$ the set of gates in $\mathcal{L}_k \setminus \mathcal{L}_{k-1}$

Our main interest is in $\bar{\mathcal{L}}_2$ gates.

Defining $\tilde{U}$ as : $\langle k|\langle l|\tilde{U}|i\rangle|j\rangle := \langle j|\langle l|U|i\rangle|k\rangle$

$\mathcal{L}_2$ gates satisfy:

$$(\mathbb{1}\otimes\tilde{U})(\tilde{U}\tilde{U}^\dagger\otimes\mathbb{1})(\mathbb{1}\otimes\tilde{U}^\dagger) = \mathbb{1}\otimes\tilde{U}\tilde{U}^\dagger$$

$$(\tilde{U}\otimes\mathbb{1})(\mathbb{1}\otimes\tilde{U}\tilde{U}^\dagger)(\tilde{U}^\dagger\otimes\mathbb{1}) = \tilde{U}\tilde{U}^\dagger\otimes\mathbb{1}$$

Construction of $\bar{\mathcal{L}}_2$ gates:

For a given local dimension q, various construct-
-ions of dual unitary gates are known.
However, exhaustive construction is known only for $q=2$ (qubits) [Bertini, Kos, Prosen PRL 2019]

For $\bar{\mathcal{L}}_2$ gates some constructions have been recently proposed [Yu, Wang, Kos Quantum 2024; Rampp, Rather, Claeys 2023]

As in the dual unitary case, exhaustive construction of $\bar{\mathcal{L}}_2$ gates is known only in the qubit case.

Insights about the construction and existence of hierarchical gates is provided by Schmidt decomposition:

$$U = \sum_{m=0}^{R} \lambda_m A_m \otimes B_m \, ; \quad \lambda_0 \geq \lambda_1 \geq \cdots \lambda_R > 0$$

R: Schmidt rank of U. $R \in \{1, 2, \cdots, q^2\}$

$$\mathrm{tr}(A_m^\dagger A_{m'}) = \mathrm{tr}(B_m^\dagger B_{m'}) = \delta_{mm'}$$

Special gates :-

(i) Local gates ; $\lambda_0 = q \Rightarrow U = q \dfrac{u_0}{\sqrt{q}} \otimes \dfrac{v_0}{\sqrt{q}}$

$u_0, v_0 \in \mathcal{U}(q)$ → unitary group of $q \times q$ matrices.

$\boxed{R = 1}$ (analogous to product states)

(ii) Maximally entangled gates:

$$\lambda_0 = \lambda_1 = \cdots = \lambda_{q^2-1} = 1$$

(analogous to Bell states)

$\boxed{R = q^2}$ $\quad U = \sum_{k=0}^{q^2-1} A_k \otimes B_k$

Note:- A_ks and B_ks for these gates need not be necessarily unitary, However for $q=2$ (qubit case) A_ks and B_ks can always be chosen as unitary. For higher local dimensions it is believed/conjectured that A_ks and B_ks can always chosen to be unitary
[Brahmachari, Rajmohan, Rather, Lakshminarayan PRA 2024]

Remarkably set of maximally entangled unitaries is same as dual unitary gates:

Maximally entangled unitary $\equiv$ Dual unitary
[Rather, Aravinda, Lakshminarayan PRL 2020]

Is Schmidt decomposition of $\bar{\mathcal{L}}_2$ gates also special?

From definition of $\bar{\mathcal{L}}_2$ gates it can be easily shown that Schmidt rank of $\bar{\mathcal{L}}_2$ gates cannot be maximal; $R < q^2$.

For $\bar{\mathcal{L}}_2$ gates Schmidt spectrum; $\{\lambda_m\}_{m=1}^{R}$ is always flat; $\lambda_0 = \lambda_1 = \dots = \lambda_R = \lambda$ [Foligno, Kos, Bertini 2023]

Unlike the dual unitary case, this condition is only necessary but not sufficient.

Two-qubit $\mathcal{L}_2$ gates:

Possible Schmidt ranks for two-qubit unitaries are 1, 2, and 4 (Note that two-qubit unitary of Schmidt rank 3 does not exist)

[Kraus, Cirac PRA (2001)
Nielsen, ..., Hines, PRA (2003)]

For $\bar{\mathcal{L}}_2$ gates Schmidt rank 4 is not possible therefore, only possible Schmidt ranks are 1 (local gates) and 2.

Remark: All two-qubit entangling $\mathcal{L}_2$ gates have Schmidt rank equal 2.

Consider the two-qubit CNOT gate:-

$$CX_2 = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \sigma_x \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{array}{c|cccc} & |0\rangle|0\rangle & |0\rangle|1\rangle & |1\rangle|0\rangle & |1\rangle|1\rangle \\ \hline |0\rangle|0\rangle & 1 & 0 & 0 & 0 \\ |0\rangle|1\rangle & 0 & 1 & 0 & 0 \\ |1\rangle|0\rangle & 0 & 0 & 0 & 1 \\ |1\rangle|1\rangle & 0 & 0 & 1 & 0 \end{array}$$

Schmidt form of CX_2:

$$CX_2 = \sqrt{2}\left[|0\rangle\langle 0| \otimes \frac{\mathbb{1}}{\sqrt{2}} + |1\rangle\langle 1| \otimes \frac{\sigma_x}{\sqrt{2}}\right]$$

It is easy to see that CX_2 has Schmidt rank 2 with $\lambda_0 = \lambda_1 = \sqrt{2}$. Infact, CX_2 is "the representative" of two-qubit $\mathcal{L}_2$ gates as all such gates are locally equivalent to CX_2

$\mathcal{L}_2 \equiv \{(u_1 \otimes v_1)\, CX_2\, (u_2 \otimes v_2)\}$ where u_i & v_i are special single-qubit unitary gates.

It can be shown that the generalized CNOT gate in local dimension q defined as:

$$CX_q = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X + \dots + |q-1\rangle\langle q-1| \otimes X^{q-1}; \quad X|j\rangle = |j \oplus 1\rangle$$

addition mod q

$$= \sqrt{q} \sum_{m=0}^{q-1} |m\rangle\langle m| \otimes \frac{X^m}{\sqrt{q}},$$

is an $\bar{\mathcal{L}}_2$ gate having Schmidt rank q with $\lambda_0 = \lambda_1 = \dots = \lambda_{q-1} = \sqrt{q}$

All $\bar{\mathcal{L}}_2$ gates in a given local dimension q considered so far are permutation matrices or permutation matrices multiplied with local gates that preserve $\bar{\mathcal{L}}_2$ property.

We now focus only on permutation matrices and search for $\bar{\mathcal{L}}_2$ gates having different Schmidt ranks.

$q=3$ (qutrits):-

As all two-qubit $\bar{\mathcal{L}}_2$ gates are known, we focus on local dimension $q=3$ (qutrit)

As Schmidt rank 2 unitaries with flat Schmidt decomposition do not exist for odd local dimensions, consequently two qutrit $\bar{\mathcal{L}}_2$ gates of Schmidt rank 2 do not exist. The possible Schmidt ranks for two qutrit $\bar{\mathcal{L}}_2$ gates are $\{3, \underbrace{4, 5, 6, 7, 8}_{?}\}$

CNOT: CX_3

Consider a two-qutrit permutation matrix:

$$P(|i\rangle \otimes |j\rangle) = |a_{ij}\rangle |b_{ij}\rangle;$$

$$i, j = 0, 1, 2$$

$$P = \sum_{i,j=0}^{2} |a_{ij}\rangle |b_{ij}\rangle \langle i| \langle j|$$

We write the permutation matrix in so-called block form [Hermes, Nechita (2017)]

$$P = \sum_{i,j=0}^{q-1} |i\rangle\langle j| \otimes M_{ij},$$

$$= \left(\begin{array}{c|c|c|c} M_{0,0} & M_{0,1} & \cdots & M_{0,q-1} \\ \hline M_{10} & M_{11} & \cdots & M_{1,q-1} \\ \hline \vdots & & & \\ M_{q-1,0} & M_{q-1,1} & \cdots & M_{q-1,q-1} \end{array}\right)$$

Where M_{ij} are $q \times q$ matrices consisting of entries 0 and 1. From unitarity of P it is not hard to see that:

$$\|M_{ij}\|^2 = 0, 1, \ldots, q \quad \forall\, i, j$$

For a dual unitary permutation:

$$\|M_{ij}\|^2 = 1, \quad tr(M^{\dagger}_{ij} M_{i'j'}) = \delta_{ii'}\delta_{jj'}$$

i.e, Each M_{ij} has one non-zero element ($=1$) and M_{ij} are orthogonal.

As $\bar{\mathcal{L}}_2$ gates have flat Schmidt spectrum, therefore all orthonormal non-zero blocks must have the same norm;

$$\|M_{ij}\|^2 = \text{constant} > 0$$

The number of such non-zero blocks defines the Schmidt rank of given $\mathcal{L}_2$ permutation gate. For example, the controlled not gate; $CX_q = \sum_{k=0}^{q-1} |k\rangle\langle k| \otimes X^k$, has q orthonormal blocks each having norm equal to $\sqrt{q}$. Therefore, it's Schmidt rank is q.

As $\|M_{ij}\|^2$ is an integer for an $\bar{\mathcal{L}}_2$ gate, therefore the only possible Schmidt ranks possible for $\bar{\mathcal{L}}_2$ gates in given local dimension q are factors of q^2. Trivial factors; 1 (local permutation), q^2 (Dual permutation)

For $q=3$, the only possible Schmidt rank for $\bar{\mathcal{L}}_2$ permutation gates is 3.

For composite dimension such as $q=4, 6$ different Schmidt ranks are possible.

For $q=4$, $\bar{\mathcal{L}}_2$ permutation gates having Schmidt ranks 2, 4, and 8 are possible [Rampp, Rather, Claeys 2023]

Similarly for $q=6$, $\bar{\mathcal{L}}_2$ permutation gates having Schmidt ranks 2, 4, 6, 9, 12, and 18 can be constructed.

- $\bar{\mathcal{L}}_2$ gates beyond permutation gates:

we are not aware of an $\bar{\mathcal{L}}_2$ gate which cannot be obtained by dressing a permutation gate with single-site unitaries or phases.

Conjecture: For prime local dimensions q, the only possible Schmidt rank for entangling $\bar{\mathcal{L}}_2$ gates is q.

If the above conjecture holds, this implies that the only possible entanglement velocity,

$$v_E = \frac{\log R}{\log q^2},$$

for $\bar{\mathcal{L}}_2$ circuits is $\frac{1}{2}$ in prime dimensional qudits.

Thank You For The Attention!