

# Integrable Hadamard circuits

# Kicked Ising Model

- Alternate **classical Ising model** with **quantum kicks**

$$H_I[\mathbf{h}] = \sum_{j=1}^L [JZ_jZ_{j+1} + h_jZ_j],$$

$$H_K = b \sum_{j=1}^L X_j,$$

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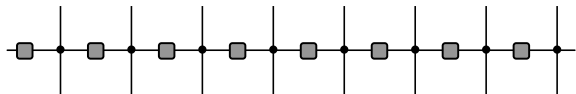
- Dual-unitary dynamics at  $|J| = |b| = \pi/4$   
... integrable at  $h_j = 0$

# Kicked Ising Model

- Alternate **controlled phases** with **one-site unitaries**

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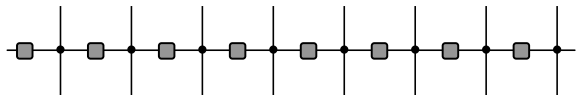
- Alternate **controlled phases** with **one-site unitaries**
- Controlled phases

$$e^{-i \sum_j J Z_j Z_{j+1} + h Z_j} \rightarrow$$


The diagram illustrates a 1D chain of qubits. A horizontal line represents the chain, with vertical lines extending upwards and downwards from each site, representing the qubit's state space. The chain is divided into segments by vertical lines. The segments alternate between controlled phases (represented by gray squares) and one-site unitaries (represented by black dots). The sequence starts with a controlled phase, followed by a one-site unitary, and so on, ending with a controlled phase.

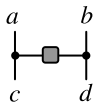
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$$e^{-i \sum_j J Z_j Z_{j+1} + h Z_j} \rightarrow$$


The diagram shows a horizontal line representing a 1D chain of qubits. There are 11 vertical lines representing qubits. Between each pair of adjacent qubits, there is a black dot on the line, representing a controlled phase gate. On each qubit line, there is a gray square representing a one-site unitary gate. The gates alternate: a one-site unitary on the first qubit, a controlled phase between the first and second qubits, a one-site unitary on the second qubit, a controlled phase between the second and third qubits, and so on, ending with a one-site unitary on the eleventh qubit.

where

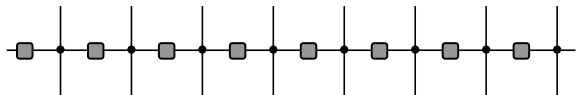


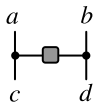
$$= e^{i\phi_{ab}} \delta_{ab} \delta_{cd} \quad \text{and} \quad \phi = \begin{pmatrix} e^{-2ih} & i \\ i & e^{2ih} \end{pmatrix}$$

The diagram shows a controlled phase gate with two input qubits, labeled 'a' (top) and 'b' (bottom), and two output qubits, labeled 'c' (top) and 'd' (bottom). A gray square gate is placed between the two qubit lines. The gate is controlled by the state of qubit 'a' and acts on qubit 'b'.

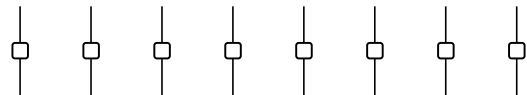
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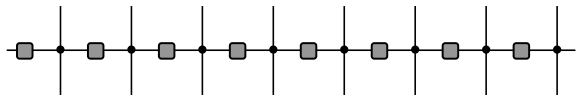
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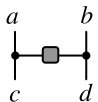
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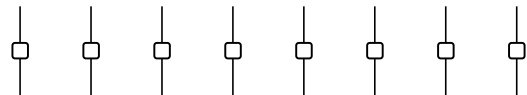
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- Alternate **controlled phases** with **one-site unitaries**
- Controlled phases

$$e^{-i \sum_j J Z_j Z_{j+1} + h Z_j} \rightarrow$$
A horizontal line representing a chain of 10 sites. Each site is marked with a black dot. Between each pair of adjacent sites, there is a vertical line representing a controlled phase. A gray square is placed on the horizontal line between each pair of sites, representing a one-site unitary. The sequence of operations is: one-site unitary, controlled phase, one-site unitary, controlled phase, one-site unitary, controlled phase, one-site unitary, controlled phase, one-site unitary, controlled phase, one-site unitary.

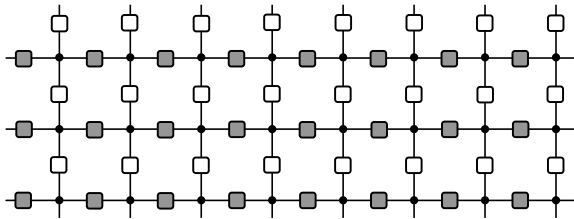
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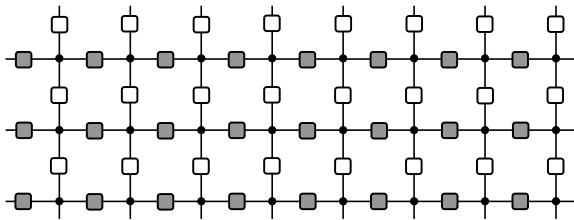
$$e^{-ib \sum_j X_j} \rightarrow$$
A horizontal line representing a chain of 8 sites. Each site is marked with a black dot. A white square with a black border is placed on the horizontal line at each site, representing a one-site unitary gate.

where   $= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$

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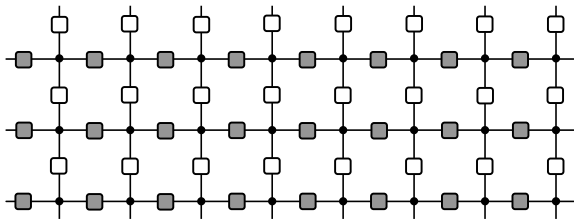
# Kicked Ising Model



- Written in terms of **delta tensors** and **complex Hadamard matrices**

$$\begin{array}{c} a \\ | \\ d - \bullet - b \\ | \\ c \end{array} = \delta_{a,b,c,d} \quad \begin{array}{c} | \\ \blacksquare \\ | \end{array} = \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \quad \begin{array}{c} | \\ \square \\ | \end{array} = \begin{pmatrix} e^{-2ih} & i \\ i & e^{2ih} \end{pmatrix}$$

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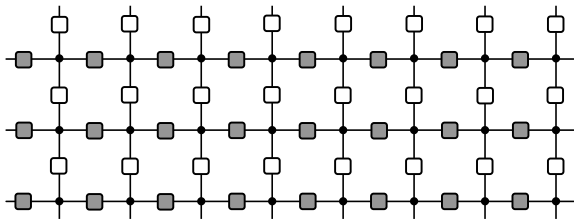
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$$HH^\dagger = H^\dagger H = q \mathbb{1} \quad |H_{ab}| = 1, \forall a, b$$

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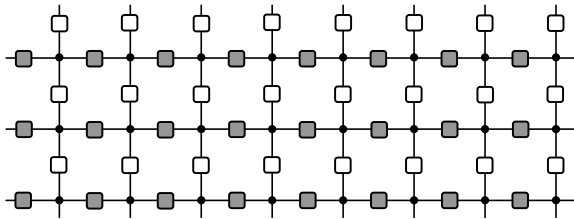
$$\begin{array}{c} | \\ \square \\ | \end{array} = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

at integrable point

- Complex Hadamard matrix:

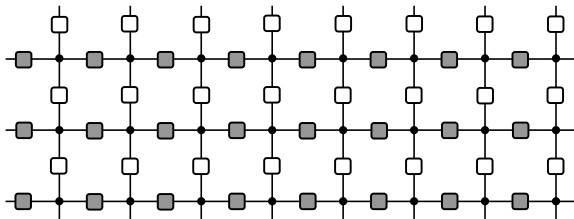
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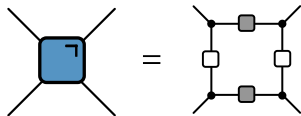


- Follows from **space-time symmetry**

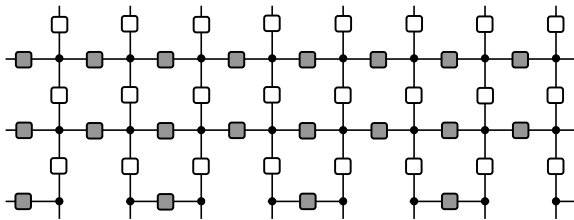
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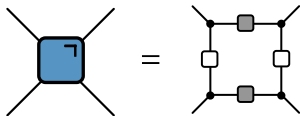
- Follows from **space-time symmetry**
- **Brickwork circuit** (with modified boundary conditions) with gates



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# Kicked Potts Model

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## Floquet integrability and long-range entanglement generation in the one-dimensional quantum Potts model

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<sup>2</sup>Alikhanov Institute for Theoretical and Experimental Physics NRC "Kurchatov Institute," Moscow 117218, Russia

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<sup>5</sup>National University of Science and Technology "MISIS," 119049 Moscow, Russia

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We develop a Floquet protocol for long-range entanglement generation in the one-dimensional quantum Potts model, which generalizes the transverse-field Ising model by allowing each spin to have  $n > 2$  states. We focus on the case of  $n = 3$ , so that the model describes a chain of qutrits. The suggested protocol creates qutrit Bell-like pairs with nonlocal long-range entanglement that spans over the entire chain. We then conjecture that the proposed Floquet protocol is integrable and explicitly construct a few first nontrivial conserved quantities that commute with the stroboscopic evolution operator. Our analysis of the Floquet integrability relies on the deep connection between the quantum Potts model and a much broader class of models described by the Temperley-Lieb algebra. We work at the purely algebraic level and our results on Floquet integrability are valid for any representation of the Temperley-Lieb algebra. We expect that our findings can be probed with present experimental facilities using Rydberg programmable quantum simulators and can find various applications in quantum technologies.

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$$H_1 = \sum_{j=1}^{2N-1} (X_j^\dagger X_{j+1} + X_j X_{j+1}^\dagger),$$

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- Expressed in terms of **clock operators**, with  $\omega = e^{2\pi i/3}$

$$X_j^3 = 1, \quad Z_j^3 = 1,$$

$$X_j^2 = X_j^\dagger = X_j^{-1}, \quad Z_j^2 = Z_j^\dagger = Z_j^{-1},$$

$$X_j Z_j = \omega Z_j X_j, \quad X_j Z_k = Z_k X_j \quad (j \neq k),$$

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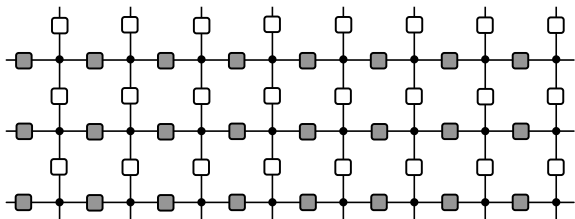
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- Floquet dynamics  $U = e^{-iH_1 T_1} e^{-iH_2 T_2}$

... integrable for  $T_1 = T_2 = \frac{2\pi}{9}(3l - m)$ ,  $m = 1, 2$

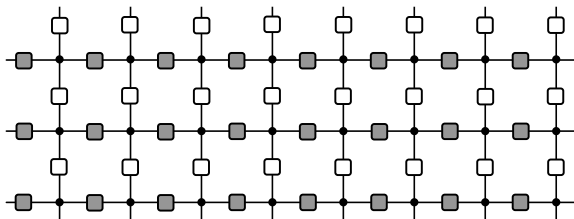
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- Shared property with KIM at integrable point

**Horizontal Hadamards = Hermitian conjugate of vertical Hadamards**

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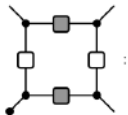
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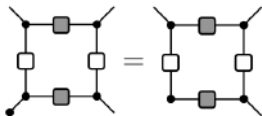
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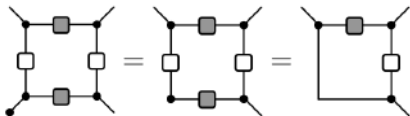
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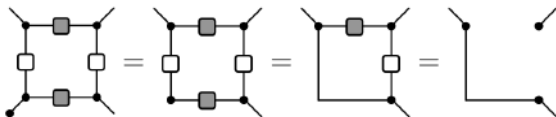
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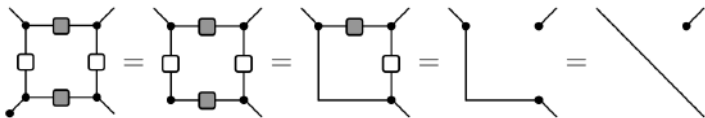
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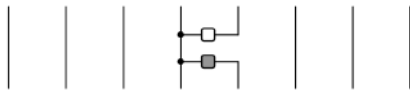
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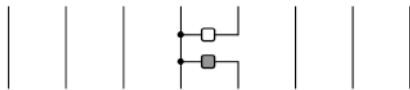
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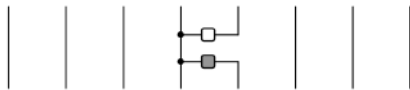
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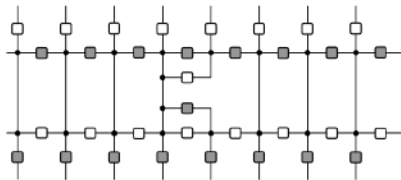
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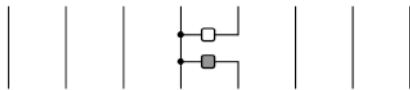


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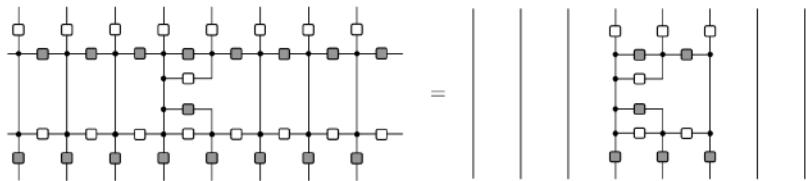


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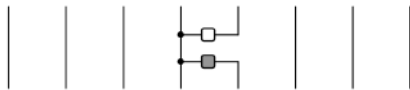


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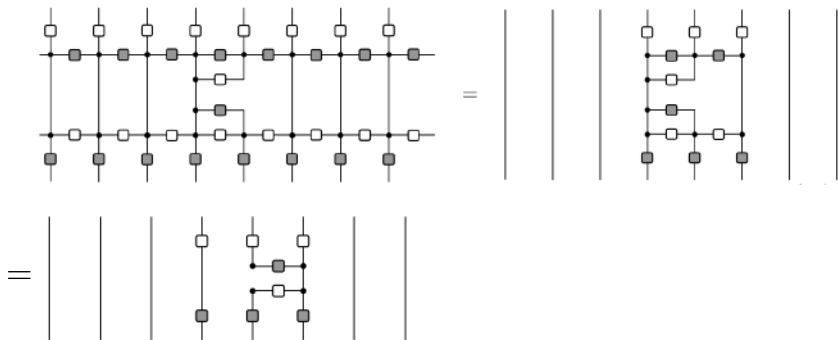


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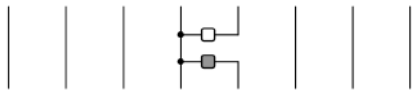


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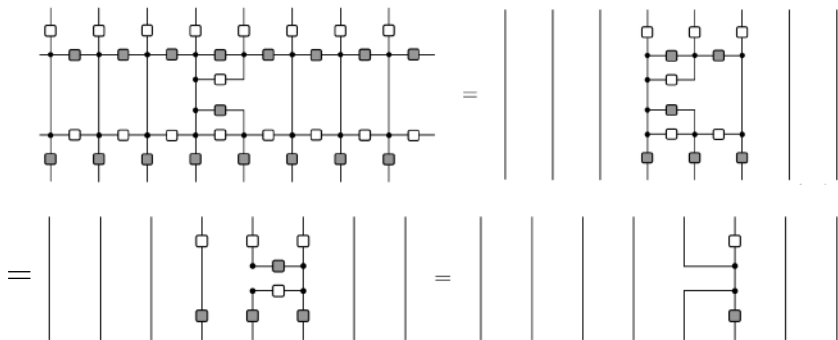


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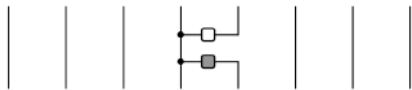


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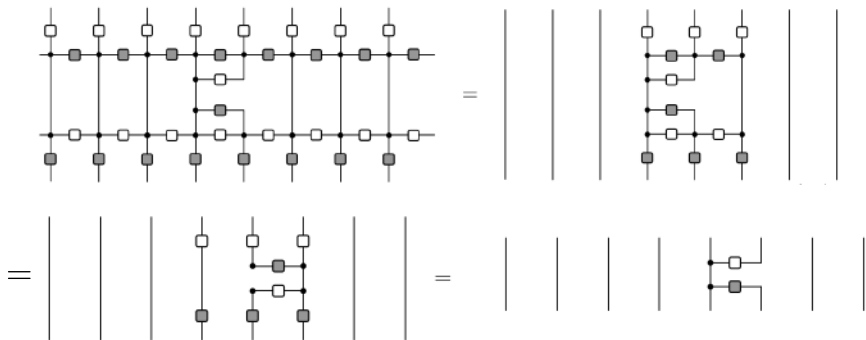


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# Gliders imply conserved charges

- Two families of **gliders**

$$\text{right moving: } q_j^{(+)} = \sum_{a=1}^d u_{\text{H}}^*(a, b) u_{\text{H}}(a, c) (|a\rangle \langle a|)_j (|b\rangle \langle c|)_{j+1}$$

$$\text{left moving: } q_j^{(-)} = \sum_{a=1}^d u_{\text{H}}^*(a, b) u_{\text{H}}(a, c) (|b\rangle \langle c|)_j (|a\rangle \langle a|)_{j+1}$$

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- Two families of **gliders**

$$\text{right moving: } q_j^{(+)} = \sum_{a=1}^d u_H^*(a, b) u_H(a, c) (|a\rangle \langle a|)_j (|b\rangle \langle c|)_{j+1}$$

$$\text{left moving: } q_j^{(-)} = \sum_{a=1}^d u_H^*(a, b) u_H(a, c) (|b\rangle \langle c|)_j (|a\rangle \langle a|)_{j+1}$$

- Resulting in **extensive set of local conserved charges**

$$Q_{k,+}^{(n_1 \dots n_{k-1})} = \sum_j q_j^{(+)} q_{j+1}^{(n_1)} \dots q_{j+k-1}^{(n_{k-1})} q_{j+k}^{(+)}, \quad n_l \in \{0, +\},$$
$$Q_{k,-}^{(n_1 \dots n_{k-1})} = \sum_j q_j^{(-)} q_{j+1}^{(n_1)} \dots q_{j+k-1}^{(n_{k-1})} q_{j+k}^{(-)}, \quad n_l \in \{0, -\},$$

with  $q_l^{(0)} = \mathbb{1}$

# Kicked Potts Model

PHYSICAL REVIEW B **105**, 144306 (2022)

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## Floquet integrability and long-range entanglement generation in the one-dimensional quantum Potts model

A. I. Lotkov<sup>1,2</sup>, V. Gritsev<sup>3,1</sup>, A. K. Fedorov<sup>1,4,5</sup> and D. V. Kurlov<sup>1</sup>


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<sup>2</sup>Alikhanov Institute for Theoretical and Experimental Physics NRC "Kurchatov Institute," Moscow 117218, Russia

<sup>3</sup>Institute for Theoretical Physics Amsterdam, Universiteit van Amsterdam, Amsterdam 1098 XH, The Netherlands

<sup>4</sup>Schaffhausen Institute of Technology, Schaffhausen 8200, Switzerland

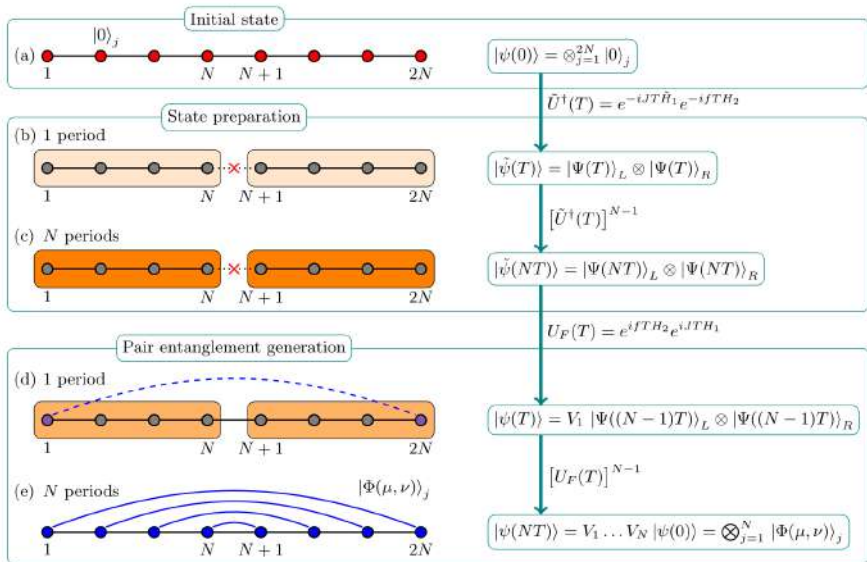
<sup>5</sup>National University of Science and Technology "MISIS," 119049 Moscow, Russia

 (Received 18 October 2021; revised 5 April 2022; accepted 8 April 2022; published 28 April 2022)

We develop a Floquet protocol for long-range entanglement generation in the one-dimensional quantum Potts model, which generalizes the transverse-field Ising model by allowing each spin to have  $n > 2$  states. We focus on the case of  $n = 3$ , so that the model describes a chain of qutrits. The suggested protocol creates qutrit Bell-like pairs with nonlocal long-range entanglement that spans over the entire chain. We then conjecture that the proposed Floquet protocol is integrable and explicitly construct a few first nontrivial conserved quantities that commute with the stroboscopic evolution operator. Our analysis of the Floquet integrability relies on the deep connection between the quantum Potts model and a much broader class of models described by the Temperley-Lieb algebra. We work at the purely algebraic level and our results on Floquet integrability are valid for any representation of the Temperley-Lieb algebra. We expect that our findings can be probed with present experimental facilities using Rydberg programmable quantum simulators and can find various applications in quantum technologies.

DOI: [10.1103/PhysRevB.105.144306](https://doi.org/10.1103/PhysRevB.105.144306)

# Entanglement generating protocol



# Entanglement generating protocol

- Initial state

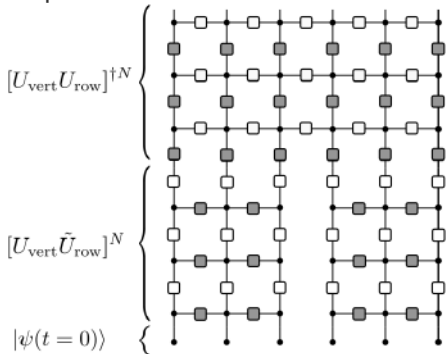
$$\bigotimes_{j=1}^{2N} |+\rangle_j = \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

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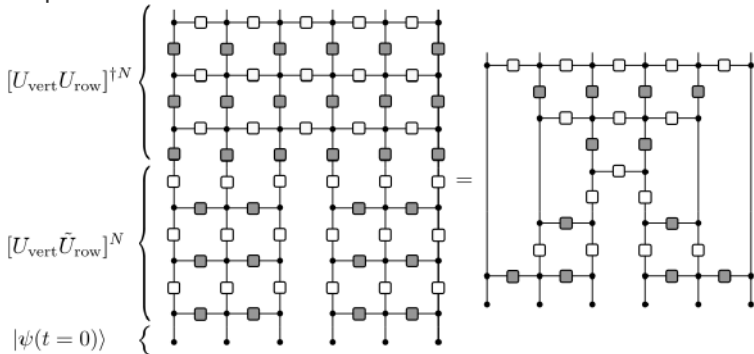


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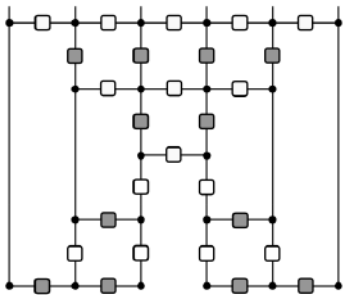


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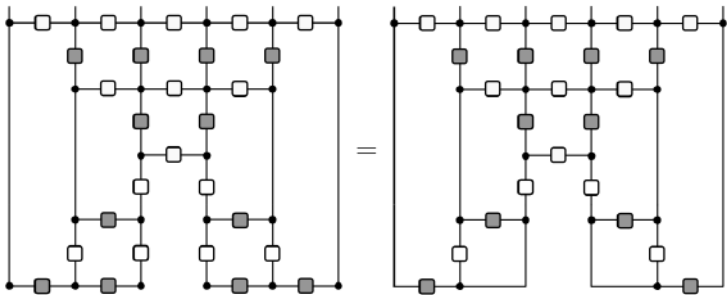


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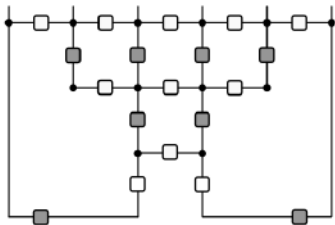


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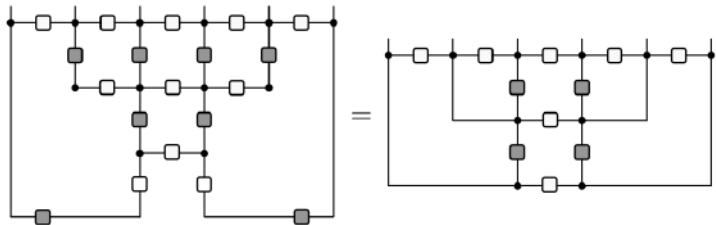


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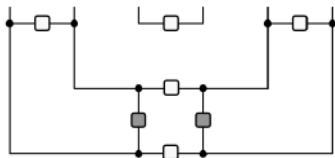


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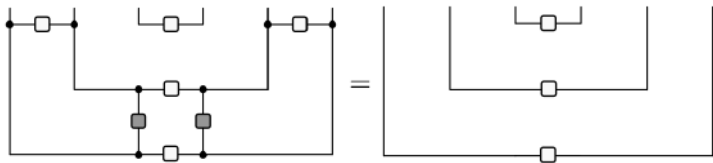


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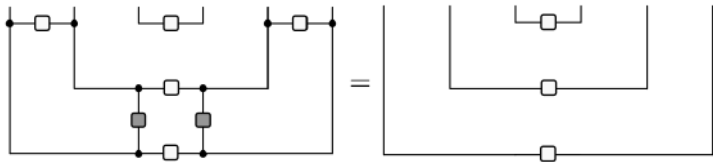


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$$|\psi_{\text{final}}\rangle = \bigotimes_{j=1}^N \left( \frac{1}{\sqrt{d}} \sum_{a,b=1}^d u_H(a,b) |a\rangle_{N-j-1} |b\rangle_{N+j} \right)$$

## Alternative approach: Fourier matrices

- General clock operators

$$Z_d = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega_d & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \omega_d^{d-1} \end{pmatrix}, \quad X_d = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

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- Fourier matrix  $F_{ab} = \omega_d^{ab}$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(N-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \omega^{3(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{bmatrix}$$

## Clifford dynamics as phase space dynamics

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$\Rightarrow$  Weyl relation  $XZ = \omega ZX$  implies **canonical commutation relation**

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**Interpretation:** Shear transformation

$$\begin{pmatrix} q \\ p \end{pmatrix} \longrightarrow \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} q \\ p + \alpha q \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

## Phase space (Clifford) dynamics

- Combine phase gate with Fourier matrix to obtain Hadamard

$$C_{jk}(\alpha, \delta) \equiv (S^\alpha F S^\delta)_{jk} = \exp\left(\frac{2\pi i}{d} \left[\frac{\alpha j^2}{2} + jk + \frac{\delta k^2}{2}\right]\right) \quad \alpha, \delta \in \mathbb{Z}.$$

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$$\text{where } \begin{pmatrix} a' \\ b' \end{pmatrix} = \overbrace{\begin{pmatrix} \alpha & \alpha\delta - 1 \\ 1 & \delta \end{pmatrix}}^{\equiv T^T} \begin{pmatrix} a \\ b \end{pmatrix} \pmod{d}$$

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- Quantum Cat Map!** (= shear + mod shift)

# Cat maps

## Arnold's cat map

🌐 8 languages ▾

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From Wikipedia, the free encyclopedia

In [mathematics](#), **Arnold's cat map** is a [chaotic](#) map from the [torus](#) into itself, named after [Vladimir Arnold](#), who demonstrated its effects in the 1960s using an image of a cat, hence the name.<sup>[1]</sup> It is a simple and pedagogical example for [hyperbolic toral automorphisms](#).

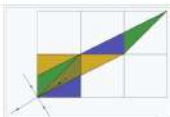
Thinking of the torus  $\mathbb{T}^2$  as the [quotient space](#)  $\mathbb{R}^2/\mathbb{Z}^2$ , Arnold's cat map is the transformation  $\Gamma: \mathbb{T}^2 \rightarrow \mathbb{T}^2$  given by the formula

$$\Gamma(x, y) = (2x + y, x + y) \bmod 1.$$

Equivalently, in [matrix](#) notation, this is

$$\Gamma \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \bmod 1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \bmod 1.$$

That is, with a unit equal to the width of the square image, the image is [sheared](#) one unit up, then two units to the right, and all that lies outside that unit square is shifted back by the unit until it is within the square.



Picture showing how the linear [cat map](#) stretches the unit square and how its pieces are rearranged when the [modulo operation](#) is performed. The lines with the arrows show the direction of the contracting and expanding [eigenspaces](#).

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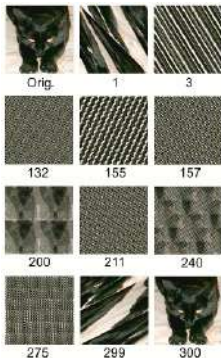
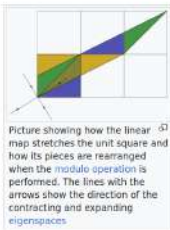
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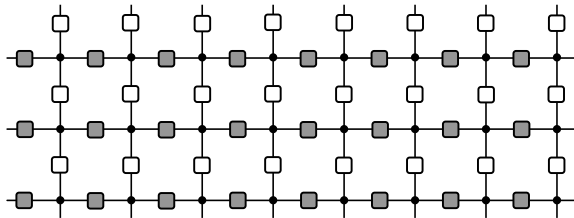
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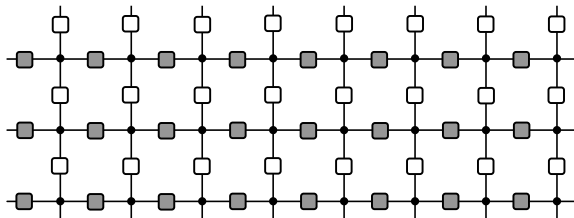


## Coupled cat maps



- **Horizontal:** Fourier matrix  $F$ , **Vertical:** Dephased Fourier matrix  $C(\alpha, \delta)$

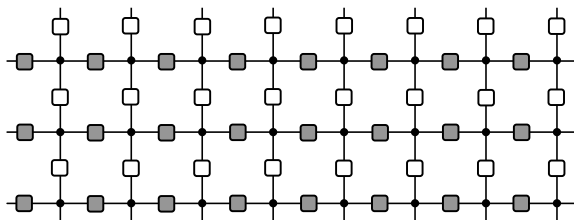
## Coupled cat maps



- **Horizontal:** Fourier matrix  $F$ , **Vertical:** Dephased Fourier matrix  $C(\alpha, \delta)$

$$\begin{aligned} a_{x,t+1} &= \alpha(a_{x,t} - b_{x-1,t} - b_{x+1,t}) + (\alpha\delta - 1)b_{x,t} \pmod{d} \\ b_{x,t+1} &= a_{x,t} - b_{x-1,t} - b_{x+1,t} + \delta b_{x,t} \end{aligned}$$

# Coupled cat maps



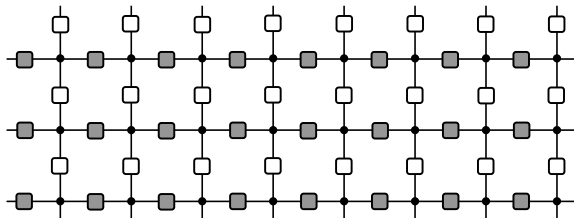
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- Eliminate single variable

$$\begin{aligned} [\Delta b]_{x,t} &= (\alpha + \delta - 4)b_{x,t} \pmod{d} \\ [\Delta b]_{x,t} &\equiv b_{x,t+1} + b_{x+1,t-1} + b_{x+1,t} + b_{x-1,t} - 4b_{x,t}. \end{aligned}$$

## Coupled cat maps



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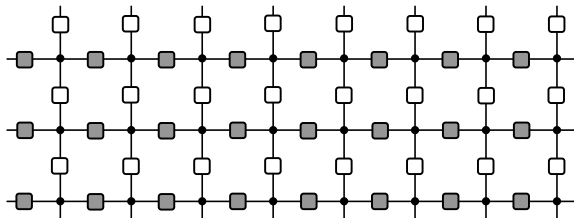
$$b_{x,t+1} = a_{x,t} + b_{x-1,t} + b_{x+1,t} + \delta b_{x,t}$$

mod  $d$

$$[\square b]_{x,t} = (\alpha + \delta)b_{x,t}$$

$$[\square b]_{x,t} \equiv b_{x,t+1} + b_{x+1,t-1} - b_{x+1,t} - b_{x-1,t}.$$

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## Gliders in coupled cat maps

$$\alpha + \delta = 0 \pmod{d}$$

- Equations of motion simplify to

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- Support **gliders**

$$\begin{aligned} a_{x,t}^R &= r_{x-t} & b_{x,t}^R &= -r_{x-t-1} \\ a_{x,t}^L &= l_{x+t} & b_{x,t}^L &= -l_{x+t+1} \end{aligned}$$

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- Fourier matrix results in **additional gliders!**

$$q_j^{(+)} = \sum_{n=0}^{d-1} Z_j^n X_{j+1}^{-n}, \quad q_j^{(-)} = \sum_{n=0}^{d-1} Z_{j+1}^n X_j^{-n}$$

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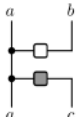
$$\begin{aligned} a_{x,t+1} &= -b_{x,t} \\ b_{x,t+1} &= a_{x,t} + b_{x-1,t} + b_{x+1,t} \end{aligned} \pmod{d}.$$

- Support **gliders**

$$\begin{aligned} a_{x,t}^R &= r_{x-t} & b_{x,t}^R &= -r_{x-t-1} \\ a_{x,t}^L &= l_{x+t} & b_{x,t}^L &= -l_{x+t+1} \end{aligned} \Rightarrow \begin{aligned} &\text{right moving: } Z_j X_{j+1}^{-1} \\ &\text{left moving: } Z_{j+1} X_j^{-1} \end{aligned}$$

- Fourier matrix results in **additional gliders!**

$$q_j^{(+)} = \sum_{n=0}^{d-1} Z_j^n X_{j+1}^{-n}, \quad q_j^{(-)} = \sum_{n=0}^{d-1} Z_{j+1}^n X_j^{-n}$$

Check:   $= F_{ab} F_{ac}^\dagger = \omega^{a(b-c)},$   $\sum_{n=0}^{d-1} Z_{aa}^n X_{bc}^{-n} = \sum_{n=0}^{d-1} \omega^{an} \delta_{b-n=c \pmod{d}} = \omega^{a(b-c)}$

## Fractals in coupled cat maps

$$\alpha + \delta \neq 0 \pmod{d}$$

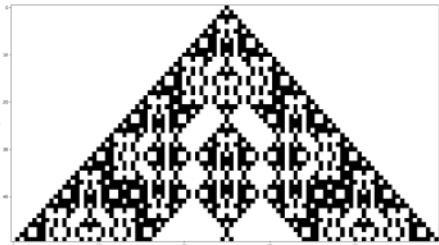
- Solutions will have **fractal structure!**

# Fractals in coupled cat maps

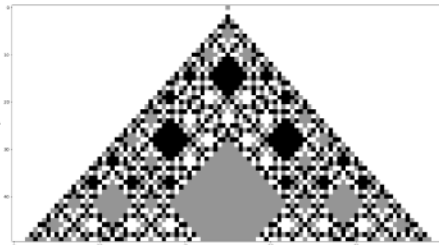
$$\alpha + \delta \neq 0 \pmod{d}$$

- Solutions will have **fractal structure!**

Kicked Ising model



Kicked Potts model

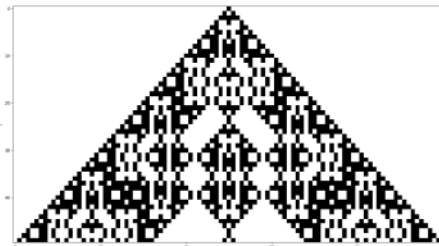


# Fractals in coupled cat maps

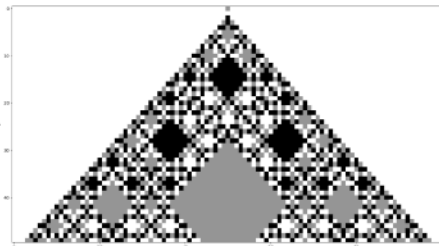
$$\alpha + \delta \neq 0 \pmod{d}$$

- Solutions will have **fractal structure!**

Kicked Ising model



Kicked Potts model



- Note that these require **Clifford dynamics!** (No operator entanglement)

Thanks for your attention!