Integrable Hadamard circuits

Kicked Ising Model

- Alternate **classical Ising model** with **quantum kicks**

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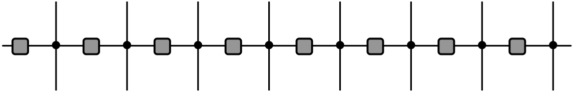
- Controlled phases

$$e^{-i\sum_j JZ_jZ_{j+1}+hZ_j} \rightarrow$$



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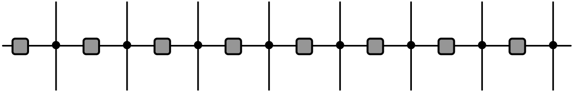
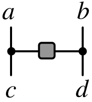


where



$$= e^{i\phi_{ab}}\delta_{ab}\delta_{cd} \quad \text{and} \quad \phi = \begin{pmatrix} e^{-2ih} & i \\ i & e^{2ih} \end{pmatrix}$$

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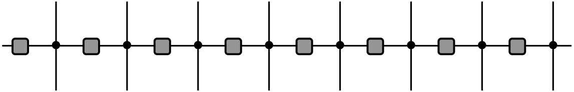

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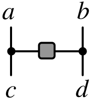

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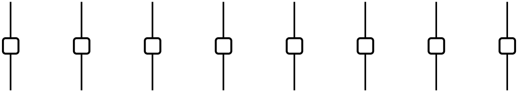

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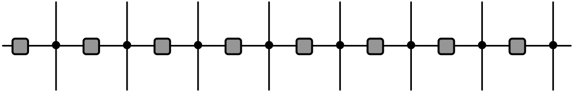
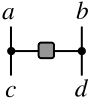
$$e^{-i\sum_j JZ_jZ_{j+1}+hZ_j} \longrightarrow$$ 

where 

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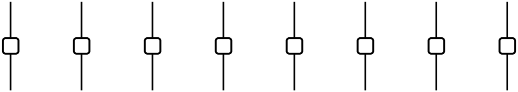
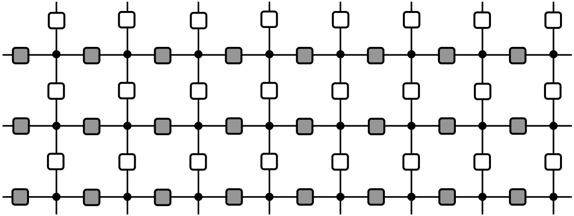
$$e^{-ib\sum_j X_j} \longrightarrow$$ 

where  $= \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$

Kicked Ising Model



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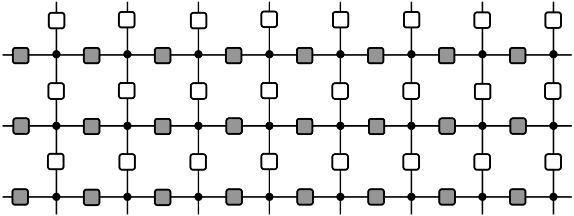
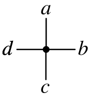


- Written in terms of **delta tensors** and **complex Hadamard matrices**



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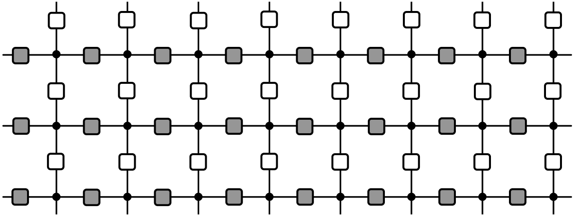

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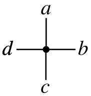

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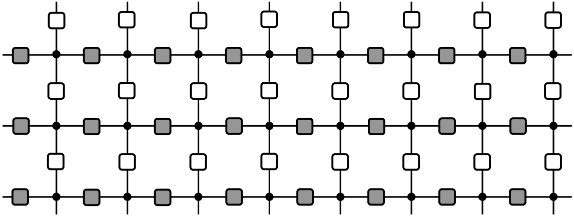

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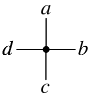

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at integrable point

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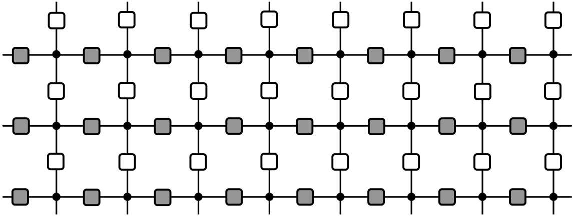

- Follows from **space-time symmetry**

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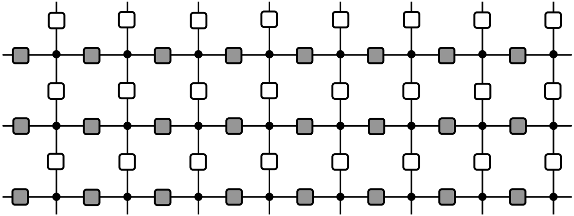

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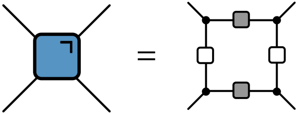

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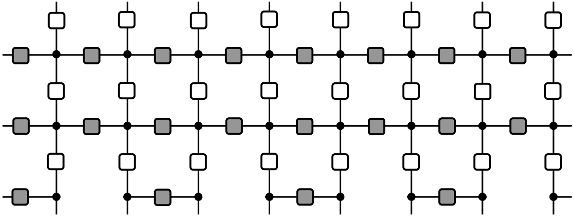

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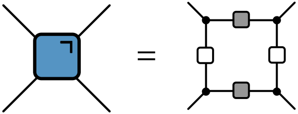

Kicked Potts Model

PHYSICAL REVIEW B **105**, 144306 (2022)

Floquet integrability and long-range entanglement generation in the one-dimensional quantum Potts model

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[2]*Alikhanov Institute for Theoretical and Experimental Physics NRC "Kurchatov Institute," Moscow 117218, Russia*
[3]*Institute for Theoretical Physics Amsterdam, Universiteit van Amsterdam, Amsterdam 1098 XH, The Netherlands*
[4]*Schaffhausen Institute of Technology, Schaffhausen 8200, Switzerland*
[5]*National University of Science and Technology "MISIS," 119049 Moscow, Russia*

(Received 18 October 2021; revised 5 April 2022; accepted 8 April 2022; published 28 April 2022)

We develop a Floquet protocol for long-range entanglement generation in the one-dimensional quantum Potts model, which generalizes the transverse-filed Ising model by allowing each spin to have $n > 2$ states. We focus on the case of $n = 3$, so that the model describes a chain of qutrits. The suggested protocol creates qutrit Bell-like pairs with nonlocal long-range entanglement that spans over the entire chain. We then conjecture that the proposed Floquet protocol is integrable and explicitly construct a few first nontrivial conserved quantities that commute with the stroboscopic evolution operator. Our analysis of the Floquet integrability relies on the deep connection between the quantum Potts model and a much broader class of models described by the Temperley-Lieb algebra. We work at the purely algebraic level and our results on Floquet integrability are valid for any representation of the Temperley-Lieb algebra. We expect that our findings can be probed with present experimental facilities using Rydberg programmable quantum simulators and can find various applications in quantum technologies.

DOI: 10.1103/PhysRevB.105.144306

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- Consider **three-state kicked Potts model**

$$H_1 = \sum_{j=1}^{2N-1} (X_j^\dagger X_{j+1} + X_j X_{j+1}^\dagger),$$

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- Expressed in terms of **clock operators**, with $\omega = e^{2\pi i/3}$

$$X_j^3 = 1, \quad Z_j^3 = 1,$$

$$X_j^2 = X_j^\dagger = X_j^{-1}, \quad Z_j^2 = Z_j^\dagger = Z_j^{-1},$$

$$X_j Z_j = \omega Z_j X_j, \quad X_j Z_k = Z_k X_j \ (j \neq k),$$

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$$Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

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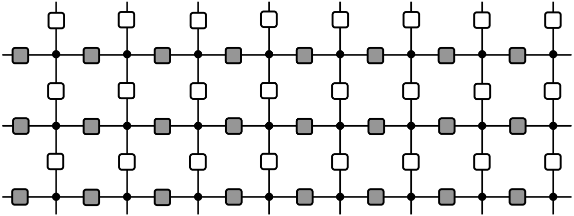
  … integrable for $T_1 = T_2 = \frac{2\pi}{9}(3l - m), \quad m = 1, 2$

Kicked Potts Model



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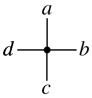

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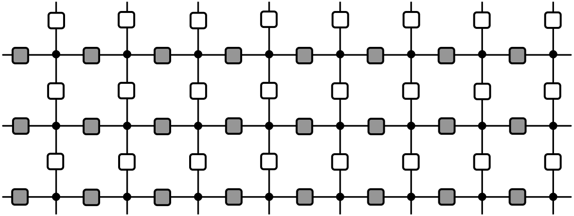

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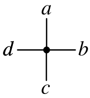

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- Shared property with KIM at integrable point

  Horizontal Hadamards = Hermitian conjugate of vertical Hadamards

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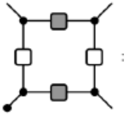

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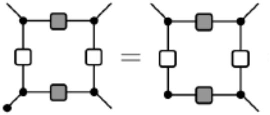

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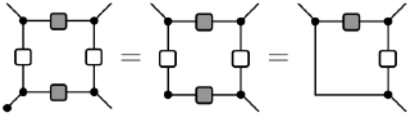

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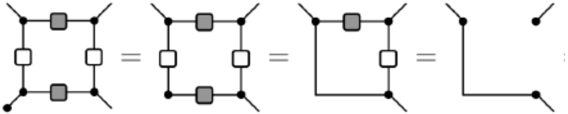

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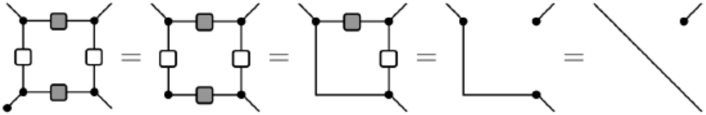
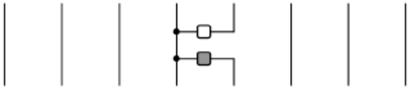


Conserved charges

- Operator dynamics supports **gliders** as dressed $\mathbb{1} \otimes |+\rangle\langle+|$



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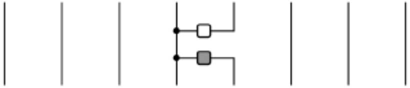

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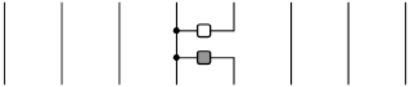

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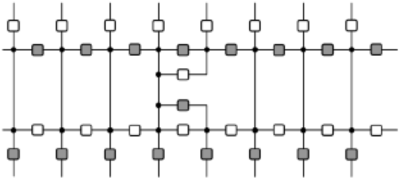

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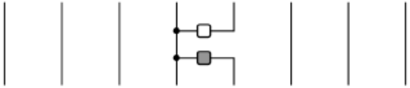

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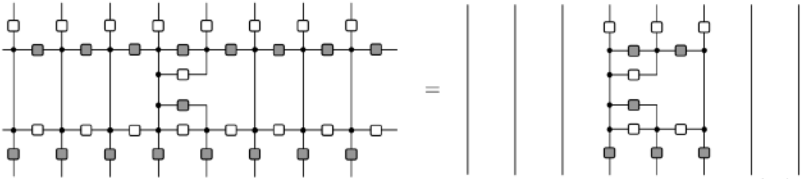

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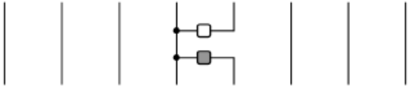

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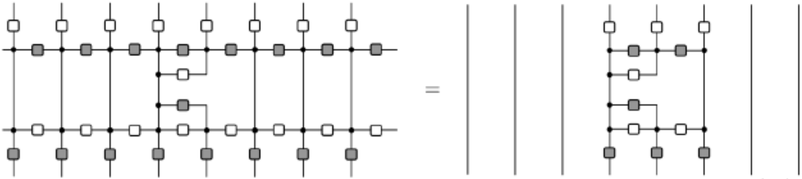

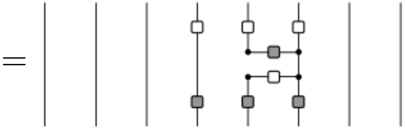

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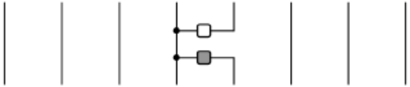

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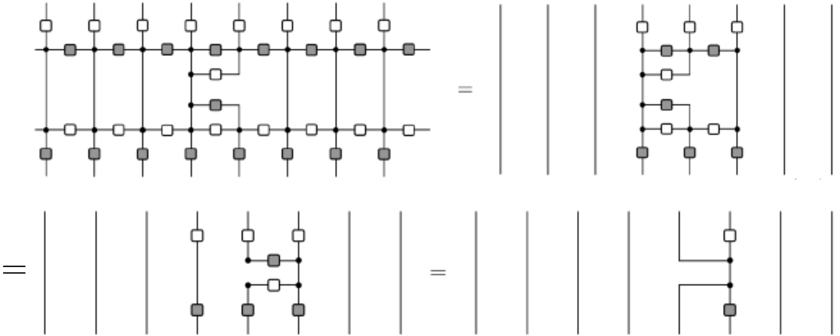

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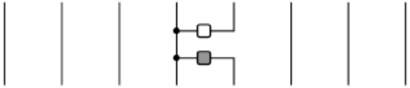

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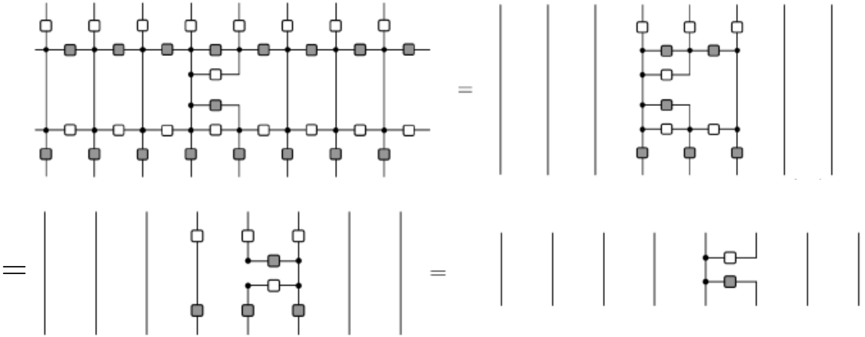

Gliders imply conserved charges

- Two families of **gliders**

$$\text{right moving: } q_j^{(+)} = \sum_{a=1}^{d} u_{\mathrm{H}}^*(a,b) u_{\mathrm{H}}(a,c) \left(|a\rangle\langle a|\right)_j \left(|b\rangle\langle c|\right)_{j+1}$$

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- Resulting in **extensive set of local conserved charges**

$$Q_{k,+}^{(n_1\ldots n_{k-1})} = \sum_j q_j^{(+)} q_{j+1}^{(n_1)} \ldots q_{j+k-1}^{(n_{k-1})} q_{j+k}^{(+)}, \qquad n_l \in \{0,+\},$$

$$Q_{k,-}^{(n_1\ldots n_{k-1})} = \sum_j q_j^{(-)} q_{j+1}^{(n_1)} \ldots q_{j+k-1}^{(n_{k-1})} q_{j+k}^{(-)}, \qquad n_l \in \{0,-\},$$

with $q_l^{(0)} = \mathbb{1}$

Kicked Potts Model

PHYSICAL REVIEW B **105**, 144306 (2022)

Floquet integrability and long-range entanglement generation in the one-dimensional quantum Potts model

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[1]*Russian Quantum Center, Skolkovo, Moscow 143025, Russia*
[2]*Alikhanov Institute for Theoretical and Experimental Physics NRC "Kurchatov Institute," Moscow 117218, Russia*
[3]*Institute for Theoretical Physics Amsterdam, Universiteit van Amsterdam, Amsterdam 1098 XH, The Netherlands*
[4]*Schaffhausen Institute of Technology, Schaffhausen 8200, Switzerland*
[5]*National University of Science and Technology "MISIS," 119049 Moscow, Russia*

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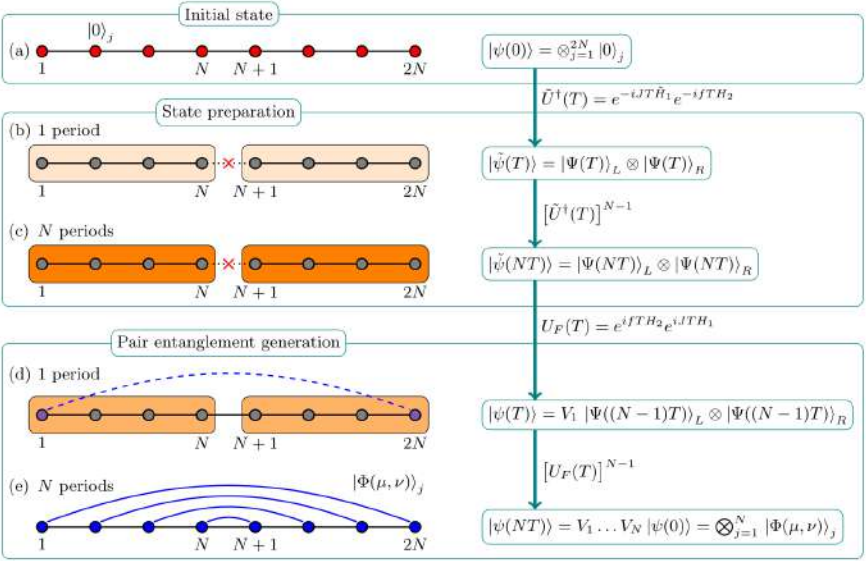
We develop a Floquet protocol for long-range entanglement generation in the one-dimensional quantum Potts model, which generalizes the transverse-filed Ising model by allowing each spin to have $n > 2$ states. We focus on the case of $n = 3$, so that the model describes a chain of qutrits. The suggested protocol creates qutrit Bell-like pairs with nonlocal long-range entanglement that spans over the entire chain. We then conjecture that the proposed Floquet protocol is integrable and explicitly construct a few first nontrivial conserved quantities that commute with the stroboscopic evolution operator. Our analysis of the Floquet integrability relies on the deep connection between the quantum Potts model and a much broader class of models described by the Temperley-Lieb algebra. We work at the purely algebraic level and our results on Floquet integrability are valid for any representation of the Temperley-Lieb algebra. We expect that our findings can be probed with present experimental facilities using Rydberg programmable quantum simulators and can find various applications in quantum technologies.

DOI: 10.1103/PhysRevB.105.144306

Entanglement generating protocol



Entanglement generating protocol

- Initial state

$$\bigotimes_{j=1}^{2N} |+\rangle_j =$$

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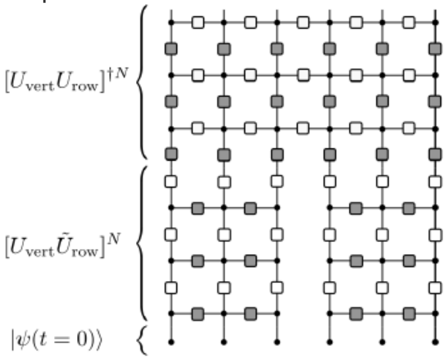

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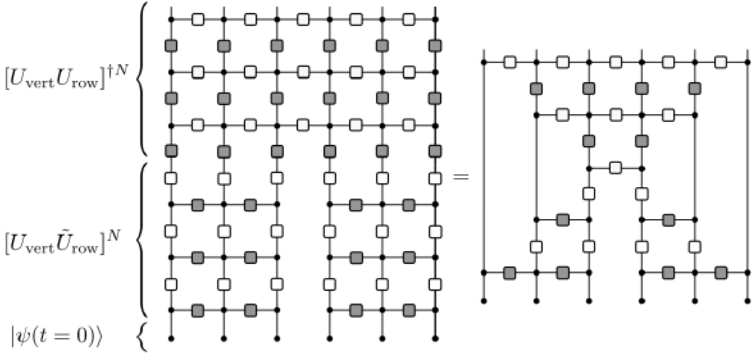

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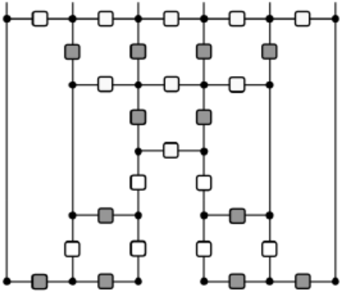

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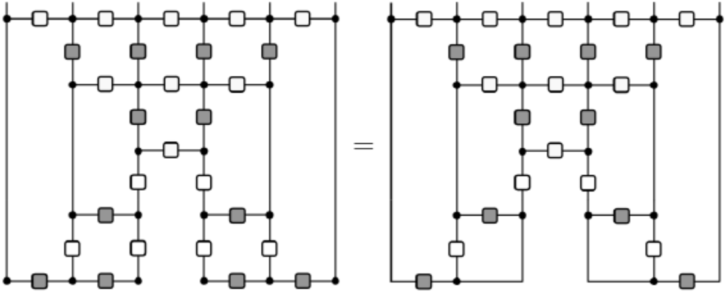

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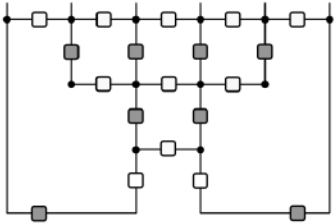

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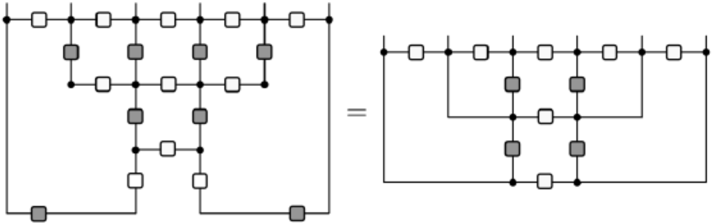

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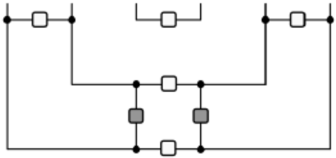

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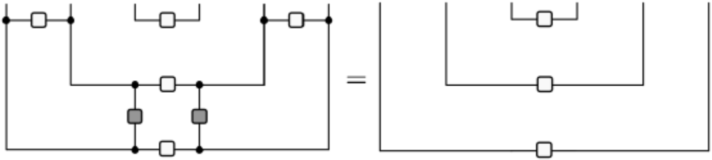

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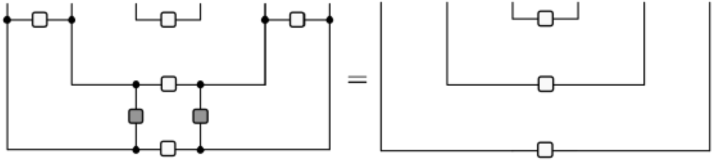

$$|\psi_{\text{final}}\rangle = \bigotimes_{j=1}^{N} \left(\frac{1}{\sqrt{d}} \sum_{a,b=1}^{d} u_{\text{H}}(a,b) \, |a\rangle_{N-j-1} \, |b\rangle_{N+j} \right)$$

Alternative approach: Fourier matrices

- General clock operators

$$Z_d = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega_d & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \omega_d^{d-1} \end{pmatrix}, \qquad X_d = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

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- Fourier matrix $F_{ab} = \omega_d^{ab}$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(N-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \omega^{3(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{bmatrix}$$

Clifford dynamics as phase space dynamics

- The Fourier matrix exchanges X and Z as

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$\Longrightarrow$ **Weyl relation** $XZ = \omega ZX$ implies **canonical commutation relation**

Phase space (Clifford) dynamics

- Unitary transformation with Z

$$\Longrightarrow Z^k Z^a X^b Z^{\dagger k} = Z^a X^b \omega^{-bk}$$

Interpretation: kick to momentum

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Interpretation: Shear transformation

$$\begin{pmatrix} q \\ p \end{pmatrix} \longrightarrow \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} q \\ p + \alpha q \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

Phase space (Clifford) dynamics

- Combine phase gate with Fourier matrix to obtain Hadamard

$$C_{jk}(\alpha,\delta) \equiv \left(S^{\alpha} F S^{\delta}\right)_{jk} = \exp\left(\frac{2\pi i}{d}\left[\frac{\alpha j^2}{2} + jk + \frac{\delta k^2}{2}\right]\right) \qquad \alpha,\delta \in \mathbb{Z}.$$

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where $$\begin{pmatrix} a' \\ b' \end{pmatrix} = \overbrace{\begin{pmatrix} \alpha & \alpha\delta - 1 \\ 1 & \delta \end{pmatrix}}^{\equiv T^T} \begin{pmatrix} a \\ b \end{pmatrix} \mod d$$

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- **Quantum Cat Map!** (= shear + mod shift)

Cat maps

Arnold's cat map

8 languages

Article Talk Read Edit View history Tools

From Wikipedia, the free encyclopedia

In mathematics, **Arnold's cat map** is a chaotic map from the torus into itself, named after Vladimir Arnold, who demonstrated its effects in the 1960s using an image of a cat, hence the name.[1] It is a simple and pedagogical example for hyperbolic toral automorphisms.

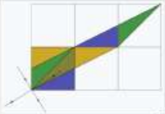
Thinking of the torus $\mathbb{T}^2$ as the quotient space $\mathbb{R}^2/\mathbb{Z}^2$, Arnold's cat map is the transformation $\Gamma : \mathbb{T}^2 \to \mathbb{T}^2$ given by the formula

$$\Gamma(x, y) = (2x + y, x + y) \bmod 1.$$

Equivalently, in matrix notation, this is

$$\Gamma\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \bmod 1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \bmod 1.$$

That is, with a unit equal to the width of the square image, the image is sheared one unit up, then two units to the right, and all that lies outside that unit square is shifted back by the unit until it is within the square.



Picture showing how the linear map stretches the unit square and how its pieces are rearranged when the modulo operation is performed. The lines with the arrows show the direction of the contracting and expanding eigenspaces

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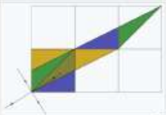

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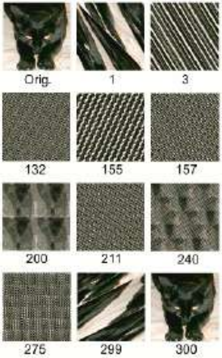
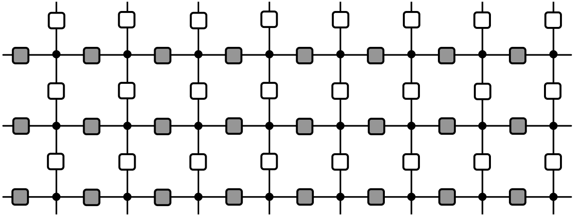


Coupled cat maps



- **Horizontal**: Fourier matrix F , **Vertical:** Dephased Fourier matrix $C(\alpha, \delta)$

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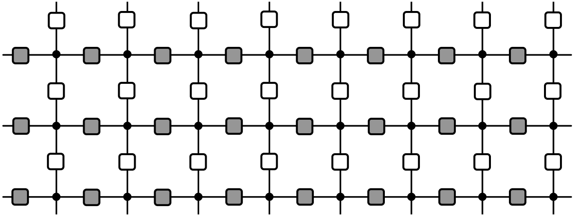

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$$\begin{aligned} a_{x,t+1} &= \alpha(a_{x,t} - b_{x-1,t} - b_{x+1,t}) + (\alpha\delta - 1)b_{x,t} \\ b_{x,t+1} &= a_{x,t} - b_{x-1,t} - b_{x+1,t} + \delta b_{x,t} \end{aligned} \quad \mod d.$$

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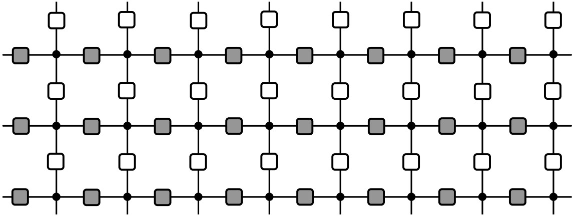

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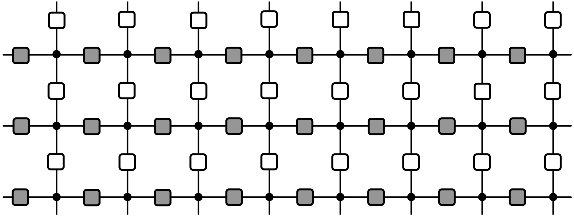
- Eliminate single variable

$$\begin{aligned} [\Delta b]_{x,t} &= (\alpha + \delta - 4)b_{x,t} \\ [\Delta b]_{x,t} &\equiv b_{x,t+1} + b_{x+1,t-1} + b_{x+1,t} + b_{x-1,t} - 4b_{x,t}. \end{aligned} \quad \mod d$$

Coupled cat maps



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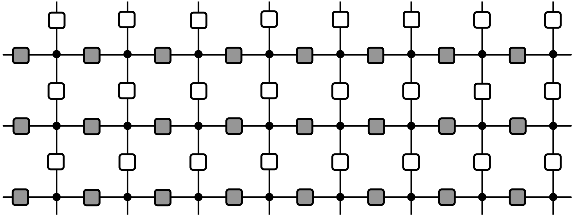

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Gliders in coupled cat maps

$$\alpha + \delta = 0 \mod d$$

- Equations of motion simplify to

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- Support **gliders**

$$\begin{aligned} a^R_{x,t} &= r_{x-t} \qquad & b^R_{x,t} &= -r_{x-t-1} \\ a^L_{x,t} &= l_{x+t} \qquad & b^L_{x,t} &= -l_{x+t+1} \end{aligned}$$

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- Fourier matrix results in **additional gliders**!

$$q_j^{(+)} = \sum_{n=0}^{d-1} Z_j^n X_{j+1}^{-n}, \qquad q_j^{(-)} = \sum_{n=0}^{d-1} Z_{j+1}^n X_j^{-n}$$

Gliders in coupled cat maps

$$\alpha + \delta = 0 \mod d$$

- Equations of motion simplify to

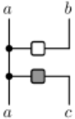
$$\begin{aligned} a_{x,t+1} &= -b_{x,t} \\ b_{x,t+1} &= a_{x,t} + b_{x-1,t} + b_{x+1,t} \end{aligned} \quad \mod d.$$

- Support **gliders**

$$\begin{aligned} a^R_{x,t} &= r_{x-t} \qquad b^R_{x,t} = -r_{x-t-1} \\ a^L_{x,t} &= l_{x+t} \qquad b^L_{x,t} = -l_{x+t+1} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} &\text{right moving: } Z_j X^{-1}_{j+1} \\ &\text{left moving: } Z_{j+1} X^{-1}_j \end{aligned}$$

- Fourier matrix results in **additional gliders**!

$$q^{(+)}_j = \sum_{n=0}^{d-1} Z^n_j X^{-n}_{j+1}, \qquad q^{(-)}_j = \sum_{n=0}^{d-1} Z^n_{j+1} X^{-n}_j$$

Check: 

$= F_{ab} F^\dagger_{ac} = \omega^{a(b-c)}, \qquad \sum_{n=0}^{d-1} Z^n_{aa} X^{-n}_{bc} = \sum_{n=0}^{d-1} \omega^{an} \delta_{b-n=c \bmod d} = \omega^{a(b-c)}$

Fractals in coupled cat maps

$$\alpha + \delta \neq 0 \mod d$$

- Solutions will have **fractal structure**!

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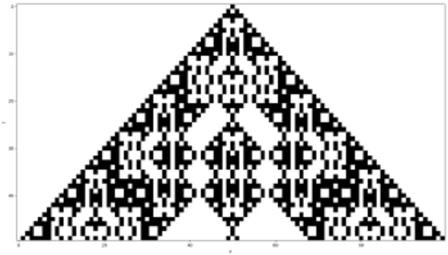
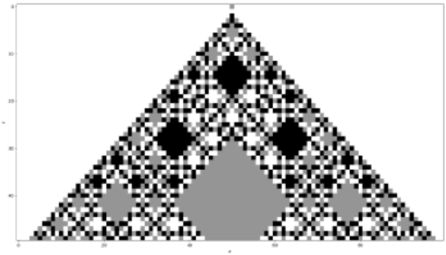
Kicked Ising model



Kicked Potts model



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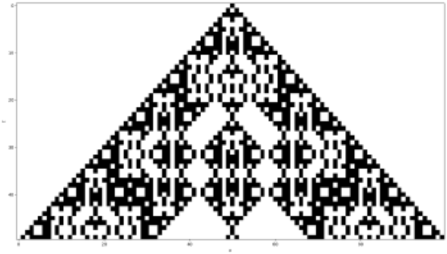
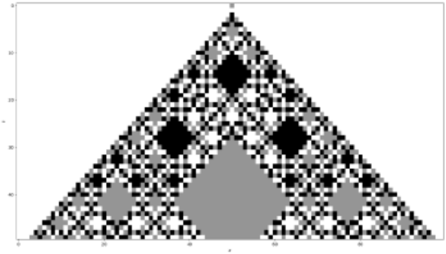


Kicked Potts model



- Note that these require **Clifford dynamics**! (No operator entanglement)

Thanks for your attention!