Magic Spreading in Random Quantum Circuits

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(1)

1) Non-Stabilizerness / Magic

- d-level systems with local Hilbertspace $\mathcal{H}_1 \cong \mathbb{C}^d$
  with basis $\{|m\rangle\}_{m \in \mathbb{Z}_d}$, (take d prime)
  where $\mathbb{Z}_d \cong \{0, \ldots, d-1\}$ is a ring with $+, \cdot$ mod d
- linear chain of N qudits $\mathcal{H}_N = \mathcal{H}_1^{\otimes d} \cong \mathbb{C}^{N^d}$
- Pauli group $P_{N,d}$: define $Z, X \in B(\mathcal{H}_1)$ via

$$Z|m\rangle = |m+1\rangle \qquad (\text{mod } d)$$
$$X|m\rangle = \omega^m |m\rangle \qquad , \omega = e^{2\pi i/d}$$

  $P_{N,d}$ generated by the local operators Z_x, X_x
  acting non-trivial as Z, X on lattice site $x \in \{1, \ldots, N\}$
- Clifford group $C_{N,d}$ (= Stabilizer of $P_{N,d}$):

$$C \in C_{N,d} < U(N^d) :\Leftrightarrow \forall P \in P_{N,d}\ \exists r \in \mathbb{Z}_d\ \exists P' \in P_{N,d}:$$
$$CPC^\dagger = \omega^r P'$$

- Stabilizer states: $\mathrm{Stab}_{N,d} = \{C|0\rangle^{\otimes N} \mid C \in C_{N,d}\}$

$\rightarrow$ magic states $|\psi\rangle \in \mathcal{H}_N \setminus \mathrm{Stab}_{N,d}$

2) Calderbank - Shor - Steane (CSS) Codes and Entropies:

- from Stabilizer Formalism
- defined on k replicas
- Defect subspace: ~~(non trivial / Submodule)~~ $\mathcal{A} \leq \mathbb{Z}_d^k$

  of ~~rank~~ / dim $0 < r_{\mathcal{A}} < k$: $\quad \forall x \in \mathcal{A}$

  i) $\quad x \cdot x = \sum_{i=1}^{k} x_i^2 = 0 \mod D \qquad D = \begin{cases} 2d & d \text{ even} \\ d & d \text{ odd} \end{cases}$

  ii) $\quad 1_k x = \sum_{i=1}^{k} x_i = 0 \mod D \qquad m_k = (m, \dots, m) \in \mathbb{Z}_d^k$

- $X_p, Z_q \in \mathcal{B}(\mathcal{H}_1^{\otimes k})$ For $q, p \in \mathbb{Z}_d^k$ defined by

$$Z_q = \bigotimes_{i=1}^{k} Z^{q_i}, \quad X_p = \bigotimes_{i=1}^{k} X^{p_i}$$

- $CSS(\mathcal{A}) = \{ Z_q X_p \mid q, p \in \mathcal{A} \}$ is a stabilizer group, i.e.

  $CSS(\mathcal{A})$ is an abelian subgroup of $P_{k,d}$ of order

  $|CSS(\mathcal{A})| = |\mathcal{A}| = d^{r_{\mathcal{A}}}$

  $\longrightarrow$ give rise to an error correcting code

- Projection onto Code Space: $\quad \leq \mathcal{H}_1^{\otimes k}$

$$Q_{\mathcal{A}} := \frac{1}{|\mathcal{A}|^2} \sum_{p,q \in \mathcal{A}} Z_q X_p \quad \in \mathcal{B}(\mathcal{H}_1^{\otimes k})$$

- CSS Entropy: $\quad |\psi\rangle \in \mathcal{H}_N, \quad \rho = |\psi\rangle\langle\psi|$

$$\mathcal{Y}_{\mathcal{A}}(|\psi\rangle) = -\ln\left(\mathcal{X}_{\mathcal{A}}(|\psi\rangle)\right)$$

$$\mathcal{X}_{\mathcal{A}}(|\psi\rangle) = |\mathcal{A}|^N \operatorname{tr}\left(Q_{\mathcal{A}}^{\otimes N} \rho^{\otimes k}\right)$$

- Properties of Y_A: (as a measure of magic)

i) $Y_A(|\psi\rangle) \geq 0$ and $Y_A(|\psi\rangle) = 0 \iff |\psi\rangle \in \mathrm{Stab}_{N,d}$

ii) $Y_A(C|\psi\rangle) = Y_A(|\psi\rangle) \quad \forall\, C \in \mathcal{C}_{N,d}$

iii) Additivity: $Y_A(|\varphi\rangle \otimes |\vartheta\rangle) = Y_A(|\varphi\rangle) + Y_A(|\vartheta\rangle)$

(defined on appropriate lattices)

- (Main) Example:

For d arbitrary, $k = D$ set $\mathcal{A}_d = \mathrm{span}\{\mathbb{1}_k\}$

$$\Rightarrow Y_{\mathcal{A}_d}(|\psi\rangle) = Y_d(|\psi\rangle) = -\ln\left(\frac{1}{d^N}\sum_{P\in\mathcal{P}_{N,d}} \langle\psi|P|\psi\rangle^D\right)$$

$d = 2 \longrightarrow$ Renyi-2-Stabilizer entropy

3) Random Quantum Circuits/Setting:

- Brickwork Circuits with open boundary conditions

$$U_t = \prod_{r=1}^{t} U^{(r)} \in \mathcal{U}(\mathcal{H}_N),\ t \dots \text{time/circuit depth}$$

where: $U^{(2k)} = \prod_{i=1}^{N/2-1} U_{2i,2i+1}$, $U^{(2k+1)} = \prod_{i=1}^{N/2} U_{2i-1,2i}$

with $U^{(a)}_{bc}$ independent and Haar distributed over $\mathcal{U}(d^2)$

- Initial State: $|\psi_0\rangle = |0\rangle^{\otimes N} \in \cancel{\mathcal{H}_N}\ \mathrm{Stab}_{N,d}$

$$|\psi(t)\rangle = U_t |\psi_0\rangle$$

- Dynamics of i) quenched average:

$$\overline{Y_d(t)} = -\mathbb{E}\left(\ln\left[\cancel{Y}_d(|\psi(t)\rangle)\right]\right)$$

$\mathbb{E}$... Haar average over the $U^{(a)}_{bc}$

ii) annealed average:

$$\tilde{Y}_d(t) = -\ln\left(\mathbb{E}\left[\cancel{Y}_d(|\psi(t)\rangle)\right]\right)$$

- numerics: $I_d(|\psi(t)\rangle)$ self averaging as $N \to \infty$

$$\Rightarrow \tilde{Y}_d(t) \simeq \overline{Y_d(t)}$$

4) Tensor-Network Representation of Annealed Averages

- Operator-to-state mapping: $\mathcal{B}(\mathcal{H}) \ni A \mapsto |A\rangle\rangle \in \mathcal{H}\otimes\mathcal{H}$

  $\longrightarrow$ superoperators: $U_t(\cdot)U_t^\dagger \mapsto U_t \otimes U_t^*$

- Annealed Average:

$$\mathbb{E}\left(\Upsilon_d[|\psi(t)\rangle]\right) = |A_d|^N \langle\langle Q_A^{\otimes N} | \,\mathbb{E}\left([U_t \otimes U_t^*]^{\otimes D}\right) | \rho_0^{\otimes D}\rangle\rangle$$

$\longrightarrow$ $\mathbb{E}\left([U_t \otimes U_t^*]^{\otimes D}\right)$ is a (non-unitary) local circuit of the same form as U_t build from local gates

$$W^{(D)} = \mathbb{E}\left[\left(U_{sc}^{(a)} \otimes [U_{sc}^{(a)}]^*\right)^{\otimes D}\right]$$
$$= \mathbb{P}^{(D)} \in \mathcal{B}\left((\mathcal{X}_1 \otimes \mathcal{X}_1)^{\otimes D}\right)$$

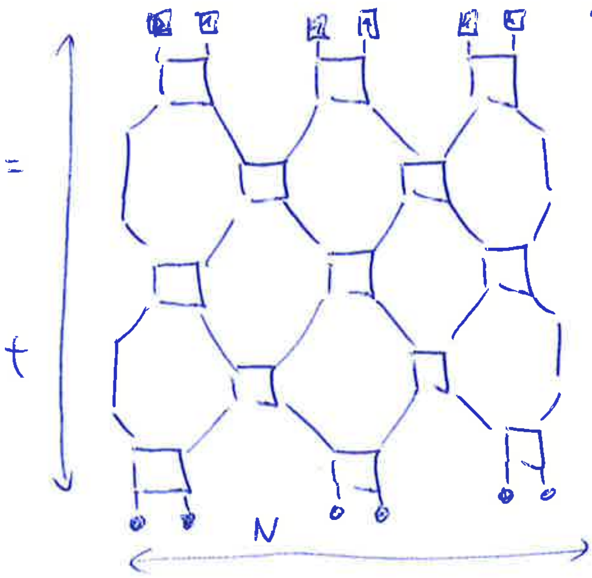
… the projection onto the subspace of $(\mathcal{H}_1 \otimes \mathcal{H}_1)^{\otimes D}$ invariant under permutations of the D replicas

… explicit expression in terms of Weingarten Functions and permutation operators/states.

- Tensor Network Representation: $\bullet = |\rho_0^{\otimes D}\rangle\rangle$, $\blacksquare = |Q_A\rangle\rangle$

  $\square = W^{(D)}$

$\mathbb{E}\left(\Upsilon_d[|\psi(t)\rangle]\right) =$



- effectively $D!$ degrees of freedom ($\pi \in S_t$) after appropriate treatment of boundary condition
- efficient contraction in horizontal direction via MPO methods

5) Numerical results:

- Saturation value at large t:

replace U_t by Haar random $U \in U(N^d)$

$$Y_2^{Haar} = -\ln\left(\frac{4}{2^N+3}\right) \qquad d=2$$

$$Y_d = -\ln\left(\frac{d^N + \sum_{k=2}^{D} C(D,k)\, d^{(k-2)N}}{(d^N+1)\cdots(d^N+D-1)}\right) \qquad d \geq 3$$

$C(D,k)$: # of permutations $\in S_D$ with exactly k cycles

- Approach to saturation (numerical for N up to 1000, $d=2,3$)

$$\tilde{Y}_d(t) = Y^{Haar} - a_d N e^{-\alpha_d t} \qquad \text{for some constants } a_d, \alpha_d$$

$\rightarrow$ time scales for saturation $t_{sat} \sim \ln(N)$ (faster than entanglement!)