# Quantum Thermodynamics and Entropy Production

## References:

- G.T. Landi, M. Paternostro, PRR, 2021
- Thermodynamics in the Quantum Regime
- Geometric and information-theoritic aspects of Quantum thermodynamics

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**Objective**: unified picture of 2nd law + link w/ information theory

Main message: information is a resource. It is consumed, stored and (inter) converted.

Irreversibility? How come? It is an emergent process - information easily become messy irretrievable when the number of degrees of freedom is large.

We adopt: "bottom up approach"

# Entropy production in Quantum Processes

Global system + Environment picture

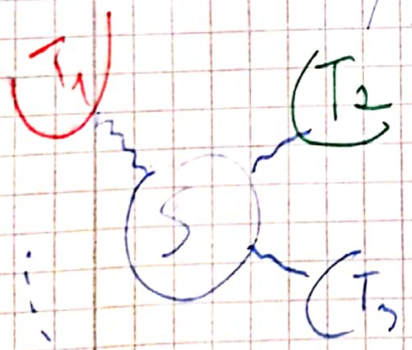
$$\rho'_{SE} = U(\rho_S \otimes \rho_E)U^\dagger$$

Local state:

$$\rho'_S = \mathcal{E}(\rho_S) = \mathrm{tr}_E\, \rho'_{SE} = \mathrm{tr}_E\{U(\rho_S \otimes \rho_E)U^\dagger\}$$

↳ Source of irreversibility! Emerges from information discarded here.

Classically one has:




- Flow of entropy: $\Phi_i = \dfrac{Q_{E_i}}{T_i}$ ($Q_{E_i} > 0$ for energy leaving)

...

~~[illegible]~~ Motivated by Clausius:

Entropy Production

$$\Sigma := \Delta S_S + \sum \frac{Q_{E_i}}{T_i} \geq 0$$

can decrease!

Note that:

> Entropy : property of the system

> " production : " of the process / transformation ;

One can also define

$$\frac{dS_S}{dt} = \dot{\Sigma} - \dot{\Phi} , \quad \dot{\Sigma} \geq 0$$

$\dot{\Phi}$ ↳ Entropy flow rate

If $\dot{\Sigma} = \dot{\Phi}$, with $\frac{dS_S}{dt} = 0$ we have a NESS.

If $\dot{\Sigma} = \dot{\Phi} = 0$ we have thermal eq.

Quantum:

$$\Sigma = I_{\rho'_{SE}}(S:E) + S(\rho'_E \| \rho_E) \quad (\square)$$

Mutual Information: (w/ $S(\rho) = \mathrm{tr}(-\rho \ln \rho)$)

$$I_{\rho_{AB}}(A:B) = S(\rho_{AB} \| \rho_A \otimes \rho_B)$$
$$= S(\rho_A) + S(\rho_B) - S(\rho_{AB})$$

(Esposito, Lindenberg, Van der Broeck - 2010)

If we no longer have access the E, this information is **lost**.

## Quantum Relative Entropy

$$S(\rho \| \sigma) = \operatorname{tr}\{\rho \ln \rho - \rho \ln \sigma\}$$

Thus, in

$S(\rho'_E \| \rho_E) \leftrightarrow$ how much we pushed E away from equilibrium

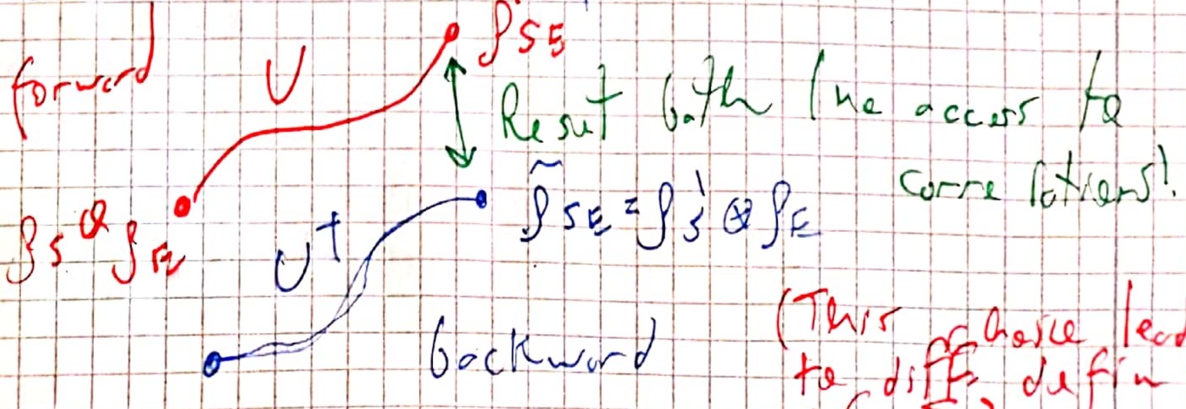
Thus:

"Both reset"

$$\Sigma = S(\rho'_{SE} \| \rho'_S \otimes \rho_E) \quad (*)$$

(Actually not unique!)

This assymetry is important in fluctuation theorems. Note that we have $\rho'_S$ (final) but $\rho_E$ (initial). Interpretation:




(This choice leads to diff. definitions of $\Sigma$)

By assuming this protocol at stochastic level we recover (*) for $\Sigma$.

Choosing a diff. protocol, i.e. a initial state $\tilde{\rho}_{SE}$ in the backward evolution leads to a different expression for $\Sigma$.

(Remember that $\Sigma = \ln \frac{P(\text{Forward})}{P(\text{Backward})}$)

— . —

Assume same splitting as classically

$$\Sigma = \Delta S_S + \Phi$$

We get:

$$\Sigma = \Delta S_S + \mathrm{tr}_E\{(\rho_E - \rho_E')\ln \rho_E\}$$

So:

$$\Phi = \mathrm{tr}_E\{(\rho_E - \rho_E')\ln \rho_E\}$$ } depends only on E!

$$\Delta S_S = S(\rho_S') - S(\rho_S)$$ } only on S! (locally)

## Maps with global fixed points

Assume (strongly?)

$$U(\rho_S^* \otimes \rho_E)U^T = \rho_S^* \otimes \rho_E$$

for a certain class of $U$

Take

$$\Phi = \mathrm{tr}_{SE}\{(\rho_S \rho_E - \rho_{SE}') \ln \rho_E\}$$

so that

$$\Phi = \mathrm{tr}_S\{(\rho_S' - \rho_S) \ln \rho_S^*\} \text{ and}$$

$$\Sigma = S(\rho_S \| \rho_S^*) - S(\rho_S' \| \rho_S^*)$$

$\Sigma$ is now a completely local quantity!

Example:

## Thermal Operations

$$[U, H_S + H_E] = 0$$

This way, $[e^{-\beta(H_S + H_E)}, U] = 0$

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$\rho_S^{th} = \frac{e^{-\beta H_S}}{Z}$ is a global fixed point.

And

$$\Sigma = S(\rho_S \| \rho_S^{th}) - S(\rho_S' \| \rho_S^{th})$$

This type of U implies:

$\Delta H_S = -\Delta H_E = Q_E$ (the energy trapped at interaction)

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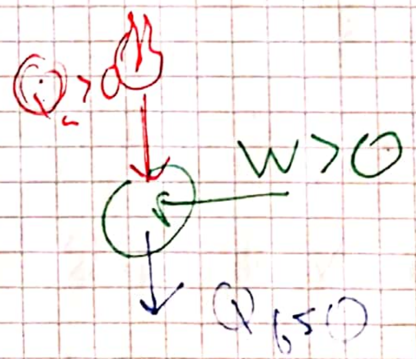
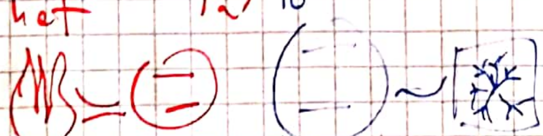
Example: Two-stroke heat engine + accelerator,



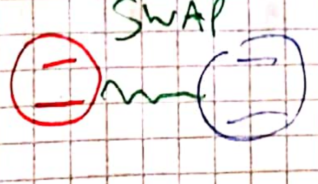

Gap in cold sink larger than in hot one

$\frac{\epsilon_b}{\epsilon_a} > 1.$

hot $T_a > T_b$



} Stroke 1




} Stroke 2

Simplest case:

(i) Thermalize to $\rho_{AB} = \rho_A^{th} \otimes \rho_B^{th}$

in stroke 1

Repeat

(ii) Change to $\rho_B^{th} \otimes \rho_A^{th}$ w/ SWAP

in stroke 2

$$W = \Delta H_A + \Delta H_B = -(\epsilon_A - \epsilon_B)(f_a - f_b)$$

w/ $f_i = (e^{\beta_i \epsilon_i} + 1)^{-1}$

[Fermi-Dirac distribution]

(iii) Go back to (i), s.t.

$$Q_a = \epsilon_a (f_a - f_b)$$

$$Q_b = \epsilon_b (f_b - f_a)$$

$\Rightarrow W + Q_a + Q_b = 0$ [cyclic process]

We have $\Delta S_s = 0$ (thermal operations)

And

$$\Sigma = -\frac{Q_A}{T_A} - \frac{Q_b}{T_b} = (-\beta_a \epsilon_a - \beta_b \epsilon_b)(f_a - f_b)$$

$$COP_h = \frac{Q_a}{W} = \frac{Q_a}{Q_b - Q_a}.$$

Since $Q_b = -W - Q_a$, we have

$$\Sigma = (\beta_b - \beta_a) Q_a + \beta_b W$$

Since $Q_a > 0$ and $W \geq 0$ for an accelerator,

$$\Sigma_{min} = (\beta_b - \beta_a) Q_a \quad \text{for } W = 0$$

Since

$COP_h = \frac{Q_a}{W}$, we have

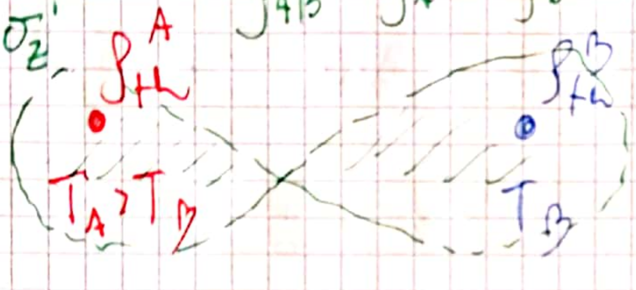
$$\boxed{COP_h = \frac{\beta_b Q_a}{\Sigma - \Sigma_{min}}}$$

The efficiency of the accelerator depends on the excess of entropy production. $\Sigma - \Sigma_{min}$.

It represents the extra irreversibility introduced by the work W pumping heat.

# Heat Flow in the presence of Correlations

$H_i = \Omega \sigma_z^i$ $\rho_{AB} = \rho_A^{th} \otimes \rho_B^{th} + \chi$ with




$$\chi = \begin{pmatrix} 0 & & & \\ & 0 & \alpha & \\ & \alpha & 0 & \\ & & & 0 \end{pmatrix}$$

denoting correlations in the Global state.

with $\rho_i^{th} = \begin{pmatrix} f_i & \\ & 1-f_i \end{pmatrix}$ with

$f_i = (e^{\Omega/T_i} + 1)^{-1}$.

Take the energy-preserving

$$U = \exp\{-igt(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+)\}$$

The heat entering $B$ is:

$$Q_B(t) = \Omega \sin(gt)[(f_A - f_B)\sin(gt) - 2\cos(gt)]$$

When $\alpha > 0$ we might have $Q_B < 0$, and heat flows from cold to hot!

# How come?

Term appearing in Eq. (D), for the entropy production $\Sigma$

Consider $\rho'_{AB} = U \rho_{AB} U^\dagger$ s.t.

$S(\rho'_{AB}) = S(\rho_{AB})$.

The change in mutual info. is

$$\Delta I_{\rho_{AB}}(A:B) = \Delta S_A + \Delta S_B$$

Consider now:

$$S_{tot} = S(\rho''_A \| \rho_A) + S(\rho''_B \| \rho_B) \geq 0,$$

which is local.

Since $\rho_A$ and $\rho_B$ are thermal:

$$S_{tot} = \beta_A \Delta H_A + \beta_B \Delta H_B - \Delta I(A:B) \geq 0$$

and

$Q_B = \Delta H_B = -\Delta H_A$ (strict energy conservation)

Thus:

$$(\beta_B - \beta_A) Q_B \geq \Delta I(A:B)$$

second law in the presence of correlations

If $\Delta I < 0$ heat can flow from hot to cold!

Correlations / mutual information are a resource!
It has to be incorporated into the 2nd law and can be consumed in thermodynamic tasks!

TL;DR: we can consume mutual info. as a quantum resource and make heat flow from cold to hot!