Physics of LDPC codes

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2411.02384
2412.13248

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Ordered phases $\leftrightarrow$ Error correction

↳ gauge theories
↳ dualities

↳ classical
↳ stabilizer CSS

Ex: Repetition code

$$|0\rangle_{\text{logical}} = |00\ldots0\rangle$$

$$|1\rangle_{\text{logical}} = |11\ldots1\rangle$$

Ordered

Ground states of $H_{\text{Ising}} = -\sum_j \sigma_j^z \sigma_{j+1}^z$

→ Commuting parity checks $\sigma_j^z \sigma_{j+1}^z$

→ n bits, 2^1 logical bits

Motivation : (good) LDPC codes

logical bits

$[[n, k, d]]$

qubits $\qquad$ code distance

Good codes : $k, d \sim n$

Repetition $[[n, 1, n]]$ → general property of Euclidean

Gauging the Ising model

$$H = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^x$$

$h \gg J$: symmetric
$h \ll J$: ~~symmetry~~ broken

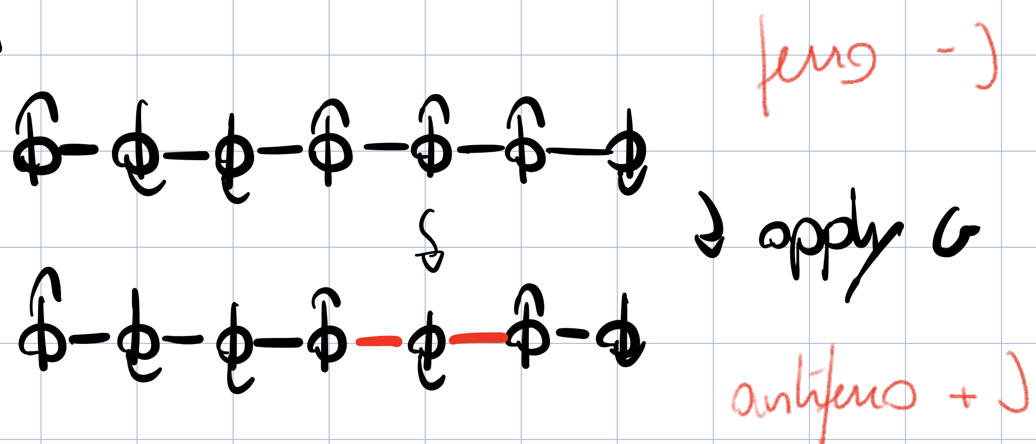
gauge field τ_{ij}

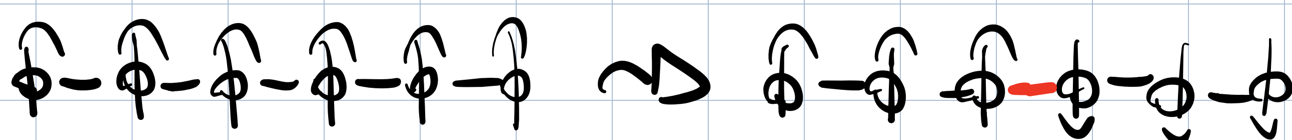
$$H = -J \sum_{\langle i,j \rangle} \sigma_i^z \tau_{ij}^z \sigma_j^z - h \sum_i \sigma_i^x$$

with Gauss law: $G_i = \sigma_i^x \prod_{j \in N(i)} \tau_{ij}^x$

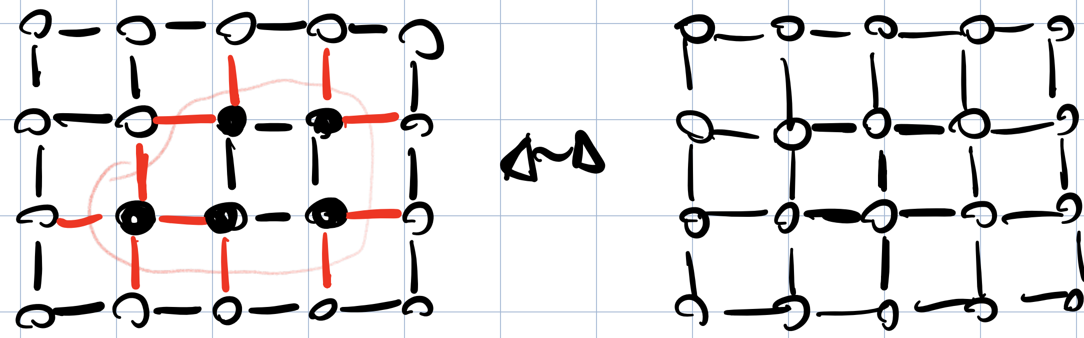
- -> commutes w/ H
- -> states related by action of G_i are physically equivalent
- -> physical subspace

1D





2D:



$\circ = \uparrow$; $\bullet = \downarrow$

$\rightsquigarrow$ Get rid of original variables, "pure gauge field"

Fix $\sigma_i^z = +1$ everywhere

$$H_{TW} = -J \sum_{\langle ij \rangle} \tau^z_{ij} - h \sum_i \prod_{\langle ij \rangle} \tau^x_{ij}$$

1D: Kramers-Wannier

So far: static fields

$$H \rightarrow H + \Gamma \sum_{\langle ij \rangle} \tau^x_{ij}$$: phase diagram?

$D \geq 2$: deconfined phase

2D : $H = -K \prod_{\square} \tau^z_{ij}$

[energetically favours "flat" configurations]
[commutes with H!]

[stable ~ toric code]

1D : unstable phase

Origin : extended vs point-like excitations

Classical LDPC codes

n spins $\sigma_i = \pm 1$, $i = 1..n$

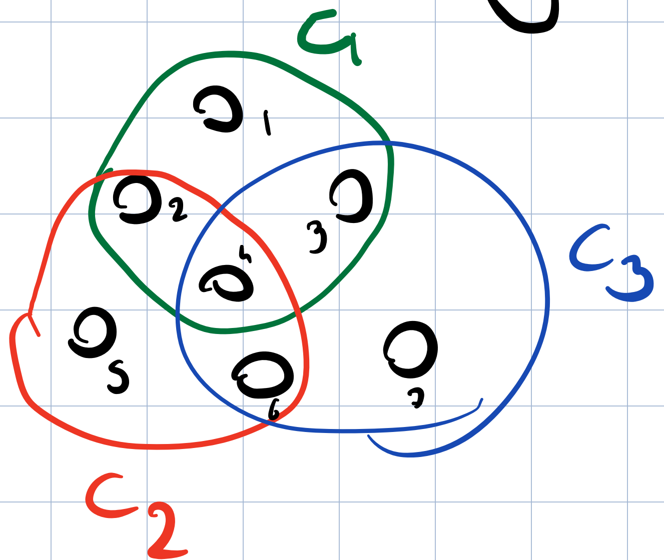
m parity checks , $C_a = \prod_{i \in S(a)} \sigma_i$

code words = configurations with $C_a = +1$ $\sim \sigma^z$

logical operators : spin flips preserving $C_a = +1$

[flip all spins associated w/ code word] $\sim \sigma^x$

Ex Hamming code $[7,4,3]$



$$C_1 = \sigma_1 \sigma_2 \sigma_3 \sigma_4$$

$$C_2 = \sigma_2 \sigma_4 \sigma_5 \sigma_6$$

$$C_3 = \sigma_3 \sigma_4 \sigma_6 \sigma_7$$

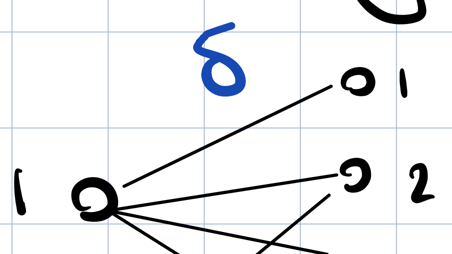
⇝ 4 codewords

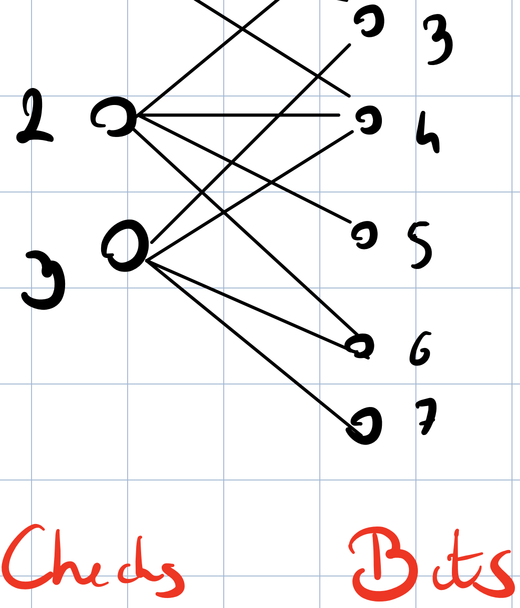
	+	+	+	+	+	+	+	(always)
	−	−	−	−	−	−	−	
	−	+	+	−	−	+	−	
Σ_i:	+	−	−	+	+	−	+	

$X_4 = \sigma_2^x \sigma_3^x \sigma_6^x$

⇝ code distance 3

⇝ Tanner graph w/ adjacency matrix





$$\delta = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

#bits × # checks

→ Codewords Σ satisfy

$$\delta^T \Sigma = 0 \pmod 2$$

→ Logicals follow from

$$\delta^T (\Sigma + \Sigma') \bmod 2$$
$$= \delta^T \Sigma + \delta^T \Sigma' \bmod 2$$
$$= 0$$

→ $\mathrm{Ker}(\delta^T)$ = Codewords / logicals
$\mathrm{Ker}(\delta)$ = Redundancies

$$\prod_{a \in R} C_a = +1 \quad \left[\text{eg Ising} : \prod_{j=1}^{n} (\sigma_j \sigma_{j+1}) = 1 \right]$$

⇝ Transpose code δ^T

* Dualities

$$C_a = \prod_{i \in \delta(a)} \sigma_i^z \; ; \; \chi_\lambda = \prod_{i \in \Sigma_\lambda} \sigma_i^x$$

checks logicals

$$A_i = \prod_{a \in \delta^T(i)} \tau_a^x \; ; \; \chi_\lambda = \prod_{a \in R_\lambda} \tau_a^z$$

transpose checks redundancies

Symmetric space: all logicals $+1$

$$\Rightarrow \dim(\mathcal{H}_\sigma^{sym}) = 2^{n-h} = 2^{m-h_a} = \dim(\mathcal{H}_i^{sym})$$

$$\Rightarrow U C_a U^\dagger = \tau_a^z \; ; \; U \sigma_i^x U^\dagger = A_i$$

$$\begin{cases} H = -J \sum_a C_a - g \sum \sigma_i^x \\ UHU^\dagger = -J \sum \tau_a^z - g \sum A_i \end{cases}$$

All commute w/ logicals?

reverses roles of checks and logicals

* Coupling to gauge fields

$$H = -J \sum_a C_a - g \sum_i \sigma_i^x$$

$$\begin{cases} C_a \to \tau_a^z C_a \\ \text{Gauss law : } G_i = \sigma_i^x A_i = \sigma_i^x \prod_{a \in S^T(i)} \tau_a^x = +1 \end{cases}$$

gauge fix $\sigma_i = +1$ returns dual Hamiltonian

* Excitations $\leftrightarrow$ structure of redundancy
  [local $\to$ plaquettes]

* Quantum LDPC
  $\to$ CSS code for n qubits

X - checks ; $A_i = \prod_{a \in S_i^T(i)} \tau_a^x$

Z - checks ; $B_p = \prod_{a \in \delta_1(p)} \tau_a^z$

$\to [A_i, B_p] = 0$

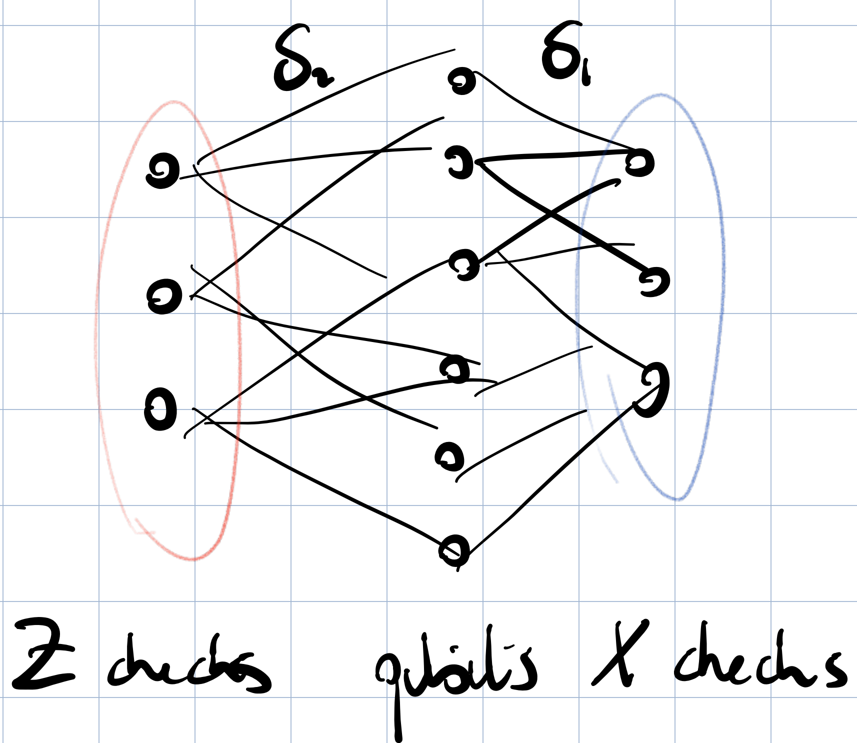
$\to$ Code subspace

$A_i |\psi\rangle = B_p |\psi\rangle = |\psi\rangle \quad \forall i, p$

$\to$ Ground states of

$$H = -\sum_i A_i - \sum_p B_p$$

[Discuss classical codes for Z, X singles at particular superposition]



Consisting : $\delta_1 \delta_2 = 0$

→ qLDPC from cLDPC

$$\begin{cases} \delta_1 = \text{classical } \delta \\ \delta_2 = \text{local redundancies of } \delta \end{cases}$$

→ Hamiltonians

$$H = -J \sum_a C_a \tau_a^z - g \sum \sigma_i^x$$
$$- K \sum_p B_p - \Gamma \sum \tau_a^x$$

Gauss : $\sigma_i^x A_i = +1$

Phase diagram ?

unitary gauge +1

$$H = -J \sum \tau_a^z - g \sum A_i - K \sum B_p$$
$$- \Gamma \sum \tau_a^x$$

→ Includes quantum CSS for $J = \Gamma = 0$!

→ Excitations

$$W_r = \prod_{a \in r} \tau_a^z \quad ; \quad H_r = \prod_{a \in \tilde{r}} \tau_a^x$$

↳ $\mathrm{Ker}(\delta_1)$ $\qquad \mathrm{Ker}(\delta_2^T)$

global?